

THE LIPSCHITZ TRUNCATION OF FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. We construct a Lipschitz truncation which approximates functions of bounded variation in the area-strict metric. The Lipschitz truncation changes the original function only on a small set similar to Lusin's theorem. Previous results could only give estimates on the Lebesgue measure of the set where the Lipschitz approximations differ from the original function.

1. INTRODUCTION

It is a classical fact attributable to LUSIN [Lus12] that any $u \in L^p(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded, open set, $1 \leq p < \infty$, can be approximated by continuous functions u_λ such that u is only changed on a small set, i.e.

$$(1.1) \quad \|u - u_\lambda\|_p \rightarrow 0 \quad \text{and} \quad \mathcal{L}^n(\{u \neq u_\lambda\}) \rightarrow 0$$

as $\lambda \rightarrow \infty$. Here, \mathcal{L}^n is the Lebesgue measure on \mathbb{R}^n . This was extended by LIU [Liu77] to Sobolev functions, showing that for any $u \in W^{1,p}(\Omega)$ one can find C^1 -approximations u_λ such that

$$(1.2) \quad \|u - u_\lambda\|_{1,p} \rightarrow 0 \quad \text{and} \quad \mathcal{L}^n(\{u \neq u_\lambda\}) \rightarrow 0$$

as $\lambda \rightarrow \infty$. This is called *Lusin property for Sobolev functions*.

A qualitative version thereof has been introduced by ACERBI & FUSCO: As established in [AF84, AF88], for any $u \in W^{1,p}(\Omega)$ with $1 \leq p < \infty$ and all $\lambda > 0$ there exist Lipschitz functions u_λ such that

$$(1.3) \quad \|\nabla u_\lambda\|_\infty \leq c\lambda, \quad \text{and} \quad \mathcal{L}^n(\{u \neq u_\lambda\}) \leq \frac{c\|u\|_{1,p}^p}{\lambda^p}$$

with c independent of u . It is possible to improve the second bound to

$$(1.4) \quad \mathcal{L}^n(\{u \neq u_\lambda\}) \leq \frac{\delta_u(\lambda)\|u\|_{1,p}^p}{\lambda^p}$$

with $\delta_u(\lambda) \rightarrow 0$ (depending on u), cf. EVANS & GARIEPY [EG92, Chapter 6.6.3, Thm. 3]. Again, this implies $\|u - u_\lambda\|_{1,p} \rightarrow 0$.

Since it is of class $W^{1,\infty}$ and coincides with u apart from a set of small Lebesgue-measure, u_λ is usually referred to as *Lipschitz truncation*. It is a core feature that u_λ differs from u only on a small set. This cannot be achieved by plain mollification.

The Lipschitz truncation has numerous applications in the calculus of variations [AF87, DLSV12], regularity theory [Lew93, CFM98, BDS16], existence of weak solutions [FMS03, DMS08, BDF12, ST19, Zha88] just to name a few.

For Lipschitz domains it is possible to preserve zero boundary data of Sobolev functions, see [Lan96]. It is possible to obtain additionally stability of the mapping $u \mapsto u_\lambda$ in all Lebesgue spaces, see [BDF12, DKS13].

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The Lipschitz truncation has been extended partially to functions of bounded variation u by EVANS & GARIEPY [EG92, Chapter. 6.6.2, Thm. 2], establishing the existence of Lipschitz approximations u_λ such (1.3) holds. However, the corresponding substitute for (1.4)

$$(1.5) \quad \mathcal{L}^n(\{u \neq u_\lambda\}) \leq \frac{\delta_u(\lambda) \|u\|_{\text{BV}(\Omega)}}{\lambda}$$

cannot be true for BV-functions. In fact, this and (1.3) would imply $u_\lambda \rightarrow u$ in $\text{BV}(\Omega)$ and thereby yield the contradictory denseness of Lipschitz functions in $\text{BV}(\Omega)$ for the norm topology; note that the respective closure of Lipschitz functions is $W^{1,1}(\Omega)$. In consequence, $\|u - u_\lambda\|_{\text{BV}} \rightarrow 0$ *cannot hold* for arbitrary BV-functions.

The goal of this paper hence is to extend the Lipschitz truncation technique to the setting of $\text{BV}(\Omega)$ with $u_\lambda \rightarrow u$ in a useful metric, necessarily weaker than the norm topology. One possibility is the notion of weak* convergence. However, this notion is too weak for many aspects. A more useful concept is the one of strict convergence, which requires that additionally the total variation converges, i.e. $|Du_\lambda|(\Omega) \rightarrow |Du|(\Omega)$.

A slight but effective modification of strict convergence is the area-strict convergence, since it is more flexible in the applications. This topology is somehow the strongest one, for which approximation by smooth functions is still possible. Moreover, the area-strict convergence (in contrast to weak* convergence) ensures both continuity of convex functionals with linear growth [Reš68] and continuity of the trace operator, cf. [EG92]. For these reasons we aim for area-strict convergence of our Lipschitz truncation.

The heart of the Lipschitz truncation is the pointwise estimate

$$(1.6) \quad |u(x) - u(y)| \leq c|x - y|(\mathcal{M}(Du)(x) + \mathcal{M}(Du)(y)) \text{ for } \mathcal{L}^n\text{-a.e. } x, y \in \mathbb{R}^n,$$

where \mathcal{M} denotes the usual Hardy-Littlewood maximal operator, being valid for any Sobolev as well as any BV-function. As such, u is Lipschitz continuous on the closed set $\mathcal{O}_\lambda^c := \{\mathcal{M}(\nabla u) \leq \lambda\}$ (the *good set*) with Lipschitz constant uniformly proportional to λ . Using a suitable extension theorem, it is possible to modify u on the *bad set* $\mathcal{O}_\lambda := \{\mathcal{M}(\nabla u) > \lambda\}$ such that its modification u_λ is Lipschitz continuous. Among all other extensions¹, the particular extension based on Whitney coverings of \mathcal{O}_λ has turned out most suitable. For this we pick a Whitney covering $(Q_j)_{j \in \mathbb{N}}$ of the bad set \mathcal{O}_λ with a corresponding partition of unity $(\eta_j)_{j \in \mathbb{N}}$. Let $(u)_{Q_j}$ denote the mean value of u over Q_j . Then the Lipschitz truncation is usually defined as

$$(1.7) \quad u_\lambda := u - \sum_{j \in \mathbb{N}} \eta_j(u - (u)_{Q_j}) = \begin{cases} u & \text{on } \mathcal{O}_\lambda^c, \\ \sum_j \eta_j(u)_{Q_j} & \text{on } \mathcal{O}_\lambda. \end{cases}$$

In particular, we replace u on the bad set locally by its mean values to obtain higher regularity. To preserve zero boundary values one has to replace $(u)_{Q_j}$ close to the boundary by zero. However, as we will see in Remark 5, this truncation does not give $u_\lambda \rightarrow u$ in the strict sense, as can be seen from

$$u : (-1, 1)^2 \rightarrow \mathbb{R}, \quad x \mapsto \text{sgn}(x_2 - x_1).$$

Here the chief issue is that the jump on the diagonal will turn into a zigzag isolines, which increases the total variation, cf. Figure 1.

On the other hand, it is well-known that mollification leads to a area-strict convergence approximation. However, this would change the function globally,

¹For some steps the McShane and the Kirszbraun extension is sufficient, but both fail the useful stability estimates, since constant functions are not necessarily extended as constant functions.

which is undesired in the applications. Thus, to overcome the problems with the classical Lipschitz truncation we propose a modified Lipschitz truncation based on local corrections using mollification. To be precise, we define

$$(1.8) \quad u_\lambda := T_\lambda u := u - \sum_{j \in \mathbb{N}} \left(\eta_j(u - (u)_{Q_j}) - \varphi_j * (\eta_j(u - (u)_{Q_j})) \right).$$

Here, for $j \in \mathbb{N}$, φ_j denotes a suitable mollifier with regularisation radius being adapted to the cube Q_j .

The main feature of the operator T_λ is that it posses a nice (almost) dual operator S_λ with

$$(1.9) \quad S_\lambda \rho := \rho - \sum_j \eta_j(\rho - \varphi_j * \rho).$$

The operator S_λ is non-expansive on L^∞ and satisfies nice commutator type estimates, see Lemma 9, i.e.

$$|\langle DT_\lambda u, \rho \rangle - \langle Du, S_\lambda \rho \rangle| \leq c h(\lambda) |Du|(\mathcal{O}_\lambda) \|\rho\|_\infty,$$

where $h(\lambda) \rightarrow 0$ for $\lambda \rightarrow \infty$.

This technique allows us to construct a modified Lipschitz truncation with $u_\lambda \rightarrow u$ in the area-strict sense, simultaneously being able to preserve zero boundary data. Our main theorem then reads as follows:

Theorem 1. *Let $\Omega = \mathbb{R}^n$ or let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain. Let $h : (0, \infty) \rightarrow (0, 1]$ be a non-increasing function with $\lim_{\lambda \rightarrow \infty} h(\lambda) = 0$. Then there exists a constant $c = c(n, \Omega) > 0$ such that for any $u \in \text{BV}(\Omega)$ and $\lambda > 0$ there is $u_\lambda : \Omega \rightarrow \mathbb{R}$ with the following properties:*

- (a) (Lipschitz property) $u_\lambda \in W^{1, \infty}(\Omega)$ together with $\|\nabla u_\lambda\|_{L^\infty(\Omega)} \leq \frac{c\lambda}{h(\lambda)^{n+1}}$.
- (b) (Small change) We have $\{u_\lambda \neq u\} \subset \mathcal{O}_\lambda := \{\mathcal{M}(Du) > \lambda\}$ and

$$\mathcal{L}^n(\mathcal{O}_\lambda) \leq c \frac{|Du|(\mathcal{O}_\lambda)}{\lambda}.$$

- (c) (Stability) The mapping $T_\lambda : u \mapsto u_\lambda$ is stable in the sense that for all $1 \leq q \leq \frac{n}{n-1}$ there holds

$$\begin{aligned} \|u_\lambda\|_{L^q(\Omega)} &\leq c \|u\|_{L^q(\Omega)}, \\ \|\nabla u_\lambda\|_{L^1(\Omega)} &\leq c |Du|(\Omega). \end{aligned}$$

- (d) (Convergence) $u_\lambda \rightarrow u$ area-strictly in $\text{BV}(\Omega)$ as $\lambda \rightarrow \infty$. More precisely, if $Du = D^a u + D^s u = \nabla u \mathcal{L}^n + D^s u$ is the Lebesgue-Radon-Nikodým decomposition of Du , then

$$(1.10) \quad \begin{aligned} \mathbb{1}_{\mathcal{O}_\lambda^c} \nabla u_\lambda &\rightarrow \nabla u && \text{in } L^1(\Omega), \\ \nabla u_\lambda \mathcal{L}^n \llcorner \mathcal{O}_\lambda &\rightarrow D^s u \llcorner \Omega && \text{area strictly,} \end{aligned}$$

as $\lambda \rightarrow \infty$. Moreover,

$$(1.11) \quad \langle DT_\lambda u \rangle(\mathbb{R}^n) \leq \langle Du \rangle(\mathbb{R}^n) + ch(\lambda) |Du|(\mathcal{O}_\lambda) + \frac{c}{\lambda} |Du|(\mathbb{R}^n),$$

$$(1.12) \quad |DT_\lambda u|(\mathbb{R}^n) \leq |Du|(\mathbb{R}^n) + ch(\lambda) |Du|(\mathcal{O}_\lambda).$$

- (e) (Zero boundary values) If $u = 0$ on Ω^c , then $u_\lambda = 0$ on Ω^c for all $\lambda \geq \lambda_0$, where $\lambda_0 = \lambda_0(n, \Omega)$.

Property (d) tells us precisely where the single parts of the approximations u_λ converge to: Namely, the restriction of the gradients Du_λ to the good set \mathcal{O}_λ^c converge to the absolutely continuous part $D^a u$, whereas the restrictions to the bad set \mathcal{O}_λ converge to the singular part $D^s u$.

The function h with $\lim_{\lambda \rightarrow 0} h(\lambda) = 0$ as it appears in (1.11) and (1.12) ensures the area strict convergence. However, in return it appears in the Lipschitz estimate in (a) additionally in the denominator.

The outline of the paper is as follows. In Section 2 we collect the requisite background facts on functions of bounded variation and maximal functions of Radon measures. Then in Section 3 we present our Lipschitz truncation for BV-functions, which concludes in Subsection 3.5 with the proof of Theorem 1.

2. PRELIMINARIES

Throughout, Ω denotes an open subset of \mathbb{R}^n with $n \geq 2$. Given $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B_r(x) := \{y \in \mathbb{R}^n : |x - y| < r\}$ the open ball of radius r centered at x . Cubes $Q \subset \mathbb{R}^n$ are always understood to be non-degenerate and parallel to the axes, and we denote $\ell(Q)$ their sidelength. The n -dimensional Lebesgue measure is denoted \mathcal{L}^n and the $n - 1$ -dimensional Hausdorff measure is denoted by \mathcal{H}^{n-1} . Sometimes we use the notation $|U| := \mathcal{L}^n(U)$ for a measurable set $U \subset \mathbb{R}^n$. We use $\mathbb{1}_U$ for the indicator function of the set U .

2.1. Radon measures. The space of \mathbb{R}^m -valued Radon measures on Ω with finite total variation is denoted $\mathcal{M}(\Omega; \mathbb{R}^m)$, i.e., $|\mu|(\Omega) < \infty$. Given $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ or $u \in L^1_{\text{loc}}(\Omega; \mathbb{R}^m)$ and a measurable subset $U \subset \Omega$ with $|U| \in (0, \infty)$, we use the notation

$$(\mu)_U := \int_U d\mu := \frac{\mu(U)}{|U|}, \quad (u)_U := \int_U u \, dx := \frac{1}{|U|} \int_U u \, dx.$$

The space $\mathcal{M}(\Omega; \mathbb{R}^m)$ can be identified with the dual space of $C_0(\Omega; \mathbb{R}^m)$. We say that μ_k converges weakly* to μ if μ_k converges weakly* in the sense of $(C_0(\Omega; \mathbb{R}^m))^*$.

Let $\mu, \mu_k \in \mathcal{M}(\Omega; \mathbb{R}^m)$. We say that μ_k converges strictly to μ if μ_k converges weakly* to μ and $|\mu_k|(\Omega) \rightarrow |\mu|(\Omega)$.

The notions of weak* convergence and strict convergence are too weak for some applications. Therefore, we introduce in the following the concepts of area-strict and f -strict convergence.

Any $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ can be decomposed as $\mu = \mu^a + \mu^s$, where $\mu^a \ll \mathcal{L}^n$ and $\mu^s \perp \mathcal{L}^n$. We shall refer to this as *Lebesgue-Radon-Nikodým decomposition* of μ .

Let $f: \mathbb{R}^m \rightarrow \mathbb{R}$ be a convex function of linear growth, i.e., there exist $c_f, C_f > 0$ such that $c_f|z| \leq f(z) \leq C_f(1 + |z|)$ holds for all $z \in \mathbb{R}^m$. We define its *recession function* $f^\infty: \mathbb{R}^m \rightarrow \mathbb{R}$ by

$$f^\infty(z) := \lim_{t \searrow 0} t f\left(\frac{z}{t}\right), \quad z \in \mathbb{R}^m.$$

Given a Radon measure $\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)$ we define the Radon measure $f(\mu)$ by

$$(2.1) \quad f(\mu) := f\left(\frac{d\mu}{d\mathcal{L}^n}\right) \mathcal{L}^n + f^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right) |\mu^s|.$$

Let $\mu, \mu_k \in \mathcal{M}(\Omega; \mathbb{R}^m)$. We say that μ_k converges f -strictly to μ if μ_k converges weakly* to μ and

$$|f(\mu_k)|(\Omega) \xrightarrow{k \rightarrow \infty} |f(\mu)|(\Omega).$$

The case $f(z) = |z|$ recovers the strict convergence.

For $f(z) = \sqrt{|z|^2 + 1}$ we abbreviate $\langle \mu \rangle := f(\mu)$. We say μ_k converges area-strictly to μ if $\langle \mu_k \rangle(\Omega) \rightarrow \langle \mu \rangle(\Omega)$. Note that the area-strict convergence of μ_k to μ is equivalent to the strict convergence of (μ_k, \mathcal{L}^n) to (μ, \mathcal{L}^n) .

2.2. Functions of bounded variation. We now collect the background definitions and facts on BV-functions, all of which can be traced back to [EG92, Chpt. 5] and [AFP00]. Let Ω be an open subset of \mathbb{R}^n . A measurable function $u: \Omega \rightarrow \mathbb{R}$ is said to be of *bounded variation* (in which case we write $u \in \text{BV}(\Omega)$) if and only if $u \in L^1(\Omega)$ and its total variation

$$|Du|(\Omega) := \sup \left\{ \int_{\Omega} u \operatorname{div}(\varphi) \, dx : \varphi \in C_c^1(\Omega; \mathbb{R}^n), |\varphi| \leq 1 \right\}$$

is finite. The norm on $\text{BV}(\Omega)$ is given by $\|u\|_{\text{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. Convergence with respect to the norm is referred to as *strong convergence*.

The Lebesgue-Radon-Nikodým decomposition of Du into its absolutely continuous and singular parts for \mathcal{L}^n reads as

$$(2.2) \quad Du = D^a u + D^s u = \nabla u \mathcal{L}^n + \frac{dD^s u}{d|D^s u|} |D^s u|,$$

where ∇u is the approximate gradient.

Given $u, u_k \in \text{BV}(\Omega)$, we say that u_k converges weakly* in $\text{BV}(\Omega)$ provided $u_k \rightarrow u$ in $L^1(\Omega)$ and $Du_k \xrightarrow{*} Du$ in $\mathcal{M}(\Omega; \mathbb{R}^n)$.

While weak* convergence is useful for compactness arguments, it is insufficient for a variety of other applications. One often needs the stronger notion of strict or area strict convergence, which we introduce in the following.

We say that u_j converges strictly (resp. area strictly or f -strictly) to u if u_j converges to u in $L^1(\Omega)$ and Du_j converges strictly to Du (resp. area strictly or f -strictly), see Subsection 2.1 for the assumptions on f .

Area-strict convergence implies f -strict convergence due to GOFFMAN & SERIN [GS64] and RESHETNYAK [Reš68]. Therefore it suffices in this article to restrict ourselves area strict convergence. Note that $u_k \rightarrow u$ in $L^1(\Omega)$ implies that

$$(2.3) \quad f(Du)(\Omega) \leq \liminf_{k \rightarrow \infty} f(Du_k)(\Omega),$$

with equality only if u_k converges to u in the f -strict sense. Area-strict convergence is in some sense the strongest topology still allowing for smooth approximation, yet being weaker than the norm topology. The single convergences are linked as follows:

$$(2.4) \quad \text{area-strict convergence} \implies \text{strict convergence} \implies \text{weak* convergence}.$$

2.3. The Hardy-Littlewood maximal operator for measures. Let us review the properties of the maximal operator on Radon measures. For a Radon measure μ on \mathbb{R}^n we define

$$(2.5) \quad \mathcal{M}\mu(x) := \sup_{Q \ni x} \mathcal{M}_Q \mu(x) := \sup_{Q \ni x} \frac{|\mu|(Q)}{\ell(Q)^n},$$

where the supremum is taken over all cubes. By the Riesz representation theorem for Radon measures, we may equivalently write

$$(2.6) \quad (\mathcal{M}\mu)(x) = \sup_{Q \ni x} \sup_{\varphi \in C_0(Q; \mathbb{R}^m) \setminus \{0\}} \frac{\langle \varphi, \mu \rangle}{\|\varphi\|_{L^\infty} \ell(Q)^n}, \quad x \in \mathbb{R}^n.$$

For future reference, we collect the most important results of the operator in

Lemma 2. *The operator \mathcal{M} as defined in (2.5) satisfies each of the following:*

- (a) *For each $\mu \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^m)$, $\mathcal{M}\mu: \mathbb{R}^n \rightarrow \mathbb{R}$ is a lower semicontinuous function.*
- (b) *There exists a constant $c = c(n) > 0$ such that $\mathcal{L}^n(\{\mathcal{M}\mu > \lambda\}) \leq \frac{c}{\lambda} |\mu|(\mathbb{R}^n)$ for all $\mu \in \mathcal{M}(\mathbb{R}^n; \mathbb{R}^m)$ and all $\lambda > 0$.*

(c) *There exists $c = c(n) > 0$ such that for any $v \in \text{BV}(\mathbb{R}^n)$ there holds*

$$|v(x) - v(y)| \leq c|x - y|(\mathcal{M}Dv(x) + \mathcal{M}Dv(y))$$

for \mathcal{L}^n -a.e. $x, y \in \mathbb{R}^n$. Here, $v(x)$ and $v(y)$ are understood in the sense of precise representatives.

Proof. Items (a) and (b) can be established completely analogous as for the well-known case of L^1 -functions. By [DS84, Thm. 2.5], for any $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ there holds

$$|u(x) - u(y)| \leq c|x - y| \left(\sup_{Q \ni x} \frac{1}{\ell(Q)} \int_Q |u - (u)_Q| dz + \sup_{Q \ni y} \frac{1}{\ell(Q)} \int_Q |u - (u)_Q| dz \right)$$

for \mathcal{L}^n -a.e. $x, y \in \mathbb{R}^n$. Now it suffices to apply the Poincaré inequality for BV-functions and the definition of \mathcal{M} to conclude the claim. The proof is complete. \square

3. LIPSCHITZ TRUNCATION IN BV

In this section we present our Lipschitz truncation of BV-functions. Let $u \in \text{BV}(\mathbb{R}^n)$ be given. If we just have $u \in \text{BV}(\Omega)$ with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$, then we can extend it by zero to all of \mathbb{R}^n using [EG92, Chapter 5.4, Theorem 1]. In this way, we then obtain that u appears as a restriction of some element of $\text{BV}(\mathbb{R}^n)$.

3.1. Whitney decomposition of the bad set. For $\lambda > 0$ we define the *bad set* $\mathcal{O}_\lambda := \{\mathcal{M}(Du) > \lambda\}$. We decompose this bad set in a standard way by means of a Whitney cover. For this we use the version [DRW10, Lemma 3.1]. We can decompose \mathcal{O}_λ into a countable family of open cubes $\{Q_j\}$, each Q_j having side length $r_j > 0$, such that the following holds:

- (W1) $\bigcup_j \frac{1}{2}Q_j = \mathcal{O}_\lambda$
- (W2) For all $j \in \mathbb{N}$ we have $8Q_j \subset \mathcal{O}_\lambda$ and $16Q_j \cap (\mathbb{R}^n \setminus \mathcal{O}_\lambda) \neq \emptyset$.
- (W3) If $Q_j \cap Q_k \neq \emptyset$, then $\frac{1}{2}r_k \leq r_j \leq 2r_k$.
- (W4) $\frac{1}{4}Q_j \cap \frac{1}{4}Q_k = \emptyset$ for all $j \neq k$.
- (W5) At every point at most 120^n of the sets $4Q_j$ intersect.

Subject to the covering $\{Q_j\}$ there exists a partition of unity $\{\eta_j\} \subset C_c^\infty(\mathbb{R}^n)$ with

- (P1) $\mathbb{1}_{\frac{1}{2}Q_j} \leq \eta_j \leq \mathbb{1}_{\frac{3}{4}Q_j}$,
- (P2) $\|\eta_j\|_\infty + r_j \|\nabla \eta_j\|_\infty + r_j^2 \|\nabla^2 \eta_j\|_\infty \leq c$.

For each $k \in \mathbb{N}$ we define $A_k := \{j : \frac{3}{4}Q_k \cap \frac{3}{4}Q_j \neq \emptyset\}$. Then

- (P3) $\sum_{j \in A_k} \eta_j = 1$ on $\frac{3}{4}Q_k$.

Moreover, we have the following:

- (W6) If $j \in A_k$, then $|Q_j \cap Q_k| \geq 16^{-n} \max\{|Q_j|, |Q_k|\}$.
- (W7) If $j \in A_k$, then $|\frac{3}{4}Q_j \cap \frac{3}{4}Q_k| \geq \max\{|Q_j|, |Q_k|\}$.
- (W8) If $j \in A_k$, then $\frac{1}{2}r_k \leq r_j < 2r_k$.
- (W9) $\#A_k \leq 120^n$.

Finally, we need the following geometric alternatives in the spirit of [DSSV17, Lemma 3.2].

Lemma 3. *Let Q be an open cube of side length r . Then one of the following alternatives holds:*

- (A1) *There exists $k \in \mathbb{N}$ such that $Q \cap \frac{1}{2}Q_k \neq \emptyset$, $8r \leq r_k$ and $Q \subset \frac{3}{4}Q_k$.*
- (A2) *For all $j \in \mathbb{N}$ with $Q \cap \frac{3}{4}Q_j \neq \emptyset$ there holds $r_j \leq 16r$ and $|Q_j| \leq 8^n |Q_j \cap Q|$. Moreover, $137Q \cap \mathcal{O}_\lambda^c \neq \emptyset$.*

Proof. If there exists $k \in \mathbb{N}$ such that $Q \cap \frac{1}{2}Q_k \neq \emptyset$ and $8r \leq r_k$, then automatically $Q \subset Q_k$. Assume now that such a k does not exist. Then for every $l \in \mathbb{N}$ with $Q \cap \frac{1}{2}Q_l \neq \emptyset$, there holds $r_l \leq 8r$. Suppose that $Q \cap \frac{3}{4}Q_j \neq \emptyset$. Now let $x \in Q \cap \frac{3}{4}Q_j$, then by (W1) there exists m such that $x \in \frac{1}{2}Q_m$. In particular, we have $Q \cap \frac{3}{4}Q_j \neq \emptyset$ due to (W2) and $\frac{1}{2}Q_m \cap \frac{3}{4}Q_j \neq \emptyset$, since both sets contain x . Now, our assumption and $Q \cap \frac{3}{4}Q_j \neq \emptyset$ imply $r_m \leq 8r$. On the other hand, $\frac{1}{2}Q_m \cap \frac{3}{4}Q_j \neq \emptyset$ and (W3) imply $r_j \leq 2r_m$. Thus, $r_j \leq 16r$. Moreover, it follows from $8r \geq r_m$ that $137Q = (1 + 17 \cdot 8)Q \supset 16Q_m$. Since $16Q_m \cap (\mathbb{R}^n \setminus \mathcal{O}_\lambda) \neq \emptyset$, we also get $137Q \cap (\mathbb{R}^n \setminus \mathcal{O}_\lambda) \neq \emptyset$. It remains to prove $|Q_j| \leq 8^n |Q_j \cap Q|$. If $r \leq \frac{1}{8}r_j$, then $Q \subset Q_j$ and the claim follows. If $r \geq \frac{1}{8}r_j$, then there exists an open cube Q' with side length $\frac{1}{8}r_j$ such that $Q' \subset Q_j \cap Q$. So in this case $|Q_j \cap Q| \geq |Q'| \geq 8^{-n} |Q_j|$. The proof is complete. \square

For each $j \in \mathbb{N}$ define

$$(3.1) \quad u_j := \begin{cases} (u)_{Q_j} = f_{Q_j} u \, dx & \text{if } \frac{3}{4}Q_j \subset \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Note that the u_j depend implicitly on λ . However, for the sake of readability we avoid an extra index.

Similar to [DKS13, Lem. 23] we obtain the following estimates for u on the Whitney cubes.

Lemma 4. *There exists a constant $c = c(n, \Omega) > 0$ such that for all $\lambda > 0$ and all $j \in \mathbb{N}$ the following holds:*

(a) *We have*

$$\int_{Q_j} \left| \frac{u - u_j}{r_j} \right| dx \leq c \frac{|Du|(Q_j)}{|Q_j|} \leq c\lambda.$$

(b) *If $k \in \mathbb{N}$ is such that $\frac{3}{4}Q_j \cap \frac{3}{4}Q_k \neq \emptyset$, then*

$$|u_k - u_j| \leq c \int_{Q_j} |u - u_j| dx + c \int_{Q_k} |u - u_k| dx.$$

(c) *If $k \in \mathbb{N}$ is such that $\frac{3}{4}Q_j \cap \frac{3}{4}Q_k \neq \emptyset$, then $|u_j - u_k| \leq cr_j \lambda$.*

Proof. Ad (a). By definition of the u_j 's, cf. (3.1), either $\frac{3}{4}Q_j \subset \Omega$, in which case we have

$$\int_{Q_j} \left| \frac{u - u_j}{r_j} \right| dx \leq c \frac{|Du|(Q_j)}{|Q_j|}$$

by Poincaré's inequality. If $\frac{3}{4}Q_j \not\subset \Omega$ we deduce that $|Q_j \cap \Omega^c| \geq c|Q_j|$ since Ω has Lipschitz boundary. Therefore, by the variant of Poincaré's inequality given in [EG92, Prop. 5.4.1],

$$\int_{Q_j} \left| \frac{u - u_j}{r_j} \right| dx = \int_{Q_j} \left| \frac{u}{r_j} \right| dx \leq c \frac{|Du|(Q_j)}{|Q_j|}.$$

By (W2), we have $16Q_j \cap \mathcal{O}_\lambda^c \neq \emptyset$ and thus find $z \in \mathcal{O}_\lambda^c$ as well as $r \leq 32r_j$ such that $16Q_j \subset B_r(z)$. Therefore,

$$\frac{|Du|(Q_j)}{|Q_j|} \leq c \frac{|Du|(16Q_j)}{|16Q_j|} \leq c \frac{|Du|(B_r(z))}{|B_r(z)|} \leq c(\mathcal{M}|Du|)(z) \leq c\lambda.$$

Ad (b). Under the assumptions of (b), we deduce from (W7) that $c \max\{|\frac{3}{4}Q_j|, |\frac{3}{4}Q_k|\} \leq |Q_j \cap Q_k|$. Thus,

$$\begin{aligned} |u_j - u_k| &\leq \int_{Q_j \cap Q_k} |u - u_j| dx + \int_{Q_j \cap Q_k} |u - u_k| dx \\ &\leq c \left(\int_{Q_j} |u - u_j| dx + \int_{Q_k} |u - u_k| dx \right) \end{aligned}$$

which implies the claim. Ad (c). By (W8), this is an immediate consequence of (a) and (b). The proof is complete. \square

3.2. Definition of the Lipschitz truncation. In this subsection we introduce a modified Lipschitz truncation. Toward Theorem 1, we begin by showing that the standard Lipschitz truncation for $W^{1,p}$ -functions cannot be employed as it does not yield strict convergence in BV. This is the content of the following remark.

Remark 5 (Failure of the standard Lipschitz truncation). *Let us explain why the standard Lipschitz truncation cannot yield strict convergence. Consider the function $u : \Omega \rightarrow \mathbb{R}$ with $\Omega = (-1, 1)^2$ and*

$$u(x) := \operatorname{sgn}(x_2 - x_1).$$

Then $u \in \operatorname{BV}(\Omega)$ and $|Du|(\Omega) = 2\sqrt{2}$. Let $\mathcal{O}_\lambda = \{\mathcal{M}(Du) > \lambda\}$ denote the bad set. Then for large λ , the set \mathcal{O}_λ and its Whitney decomposition looks roughly as in Figure 1. The standard Lipschitz truncation is defined as

$$u_\lambda := u + \sum_j \eta_j((u)_{Q_j} - u) = \begin{cases} u & \text{on } \mathcal{O}_\lambda^c, \\ \sum_j \eta_j(u)_{Q_j} & \text{on } \mathcal{O}_\lambda. \end{cases}$$

The dyadic structure of the Whitney cubes forces the isolines of u_λ (for large λ) to form a zigzag pattern, thereby increasing the length of the isolines. Hence, the co-area formula

$$|Du|(\Omega) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(u^{-1}(\{t\})) dt$$

shows that $|Du_\lambda|(\Omega) \geq (1 + \delta)|Du|(\Omega)$ for some fixed $\delta > 0$. Thus, u_λ cannot strictly converge to u in $\operatorname{BV}(\Omega)$.

On the other hand, it is well-known that mollification of a BV function yields a strictly convergent approximation. However, this approximation would differ in general from the original function almost everywhere. Therefore, we combine the standard Lipschitz truncation with a local mollification to obtain our new Lipschitz truncation converging even in the area-strict sense.

For this purpose, let $h : (0, \infty) \rightarrow (0, 1]$ be a non-increasing function with $\lim_{\lambda \rightarrow \infty} h(\lambda) = 0$. Let φ be a smooth, non-negative, radially symmetric mollifier with support in the unit ball. For $j \in \mathbb{N}$ let

$$(3.2) \quad \varphi_j(x) := (\varepsilon_j)^{-n} \varphi(x/\varepsilon_j) \quad \text{with} \quad \varepsilon_j := h(\lambda)^{\frac{1}{4}} r_j.$$

In particular, the supports of the φ_j shrink faster than the cubes Q_j by the factor of $h(\lambda)$. Furthermore, we define

$$(3.3) \quad \mathcal{B}_j u := \eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j)).$$

Then we have $\operatorname{supp}(\mathcal{B}_j) \subset Q_j$ due to $\operatorname{supp}(\eta_j) \subset \frac{3}{4}Q_j$ and the choice of ε_j . We now define our truncation operator T_λ by

$$(3.4) \quad T_\lambda u := u_\lambda := u - \sum_j \mathcal{B}_j u.$$

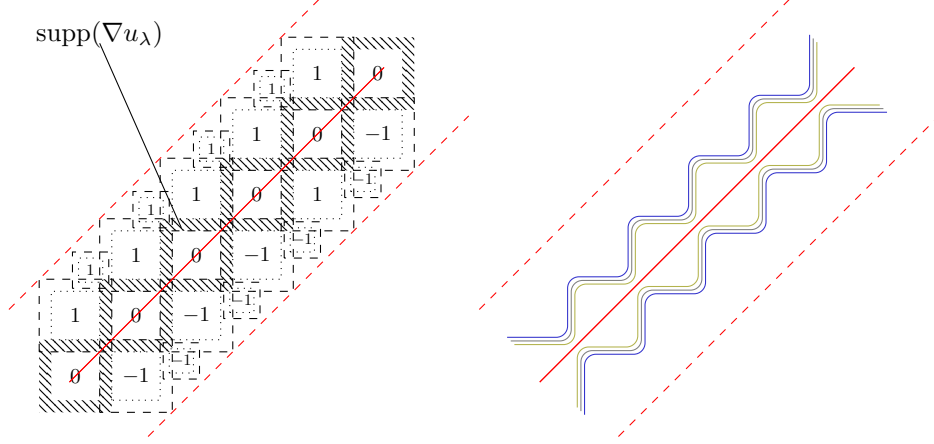


FIGURE 1. Example of Remark 5. Left: The picture shows a zoom of the Whitney cover at the diagonal. The numbers indicate the mean values $(u)_{Q_j}$. The function u_λ is locally constant outside the shaded region. Right: The picture shows the resulting isolines of u_λ .

The special choice of this truncation operator will become clear later when we consider its (almost) dual operator in Lemma 9. As we will see, the map $T_\lambda u$ defines an element in $BV(\mathbb{R}^n)$, cf. Lemma 6.

3.3. Properties of the Lipschitz truncation. In this subsection we study important properties of the Lipschitz truncation $T_\lambda u$. We begin with the stability estimates.

Lemma 6 (Stability). *There exists a constant $c = c(n) > 0$ such that we have the following L^1 - and BV-stability estimates for all $1 \leq q \leq \frac{n}{n-1}$ and all $j \in \mathbb{N}$:*

$$(3.5) \quad \|\mathcal{B}_j u\|_q \leq c \int_{Q_j} |u| \, dx \quad \text{and} \quad |D(\mathcal{B}_j u)|(\mathbb{R}^n) \leq c |Du|(Q_j).$$

The sum $\mathcal{B}_\lambda u := \sum_j \mathcal{B}_j u$ converges unconditionally in $BV(\mathbb{R}^n)$ together with

$$(3.6) \quad \|\mathcal{B}_\lambda u\|_q^q \leq c \int_{\mathcal{O}_\lambda} |u|^q \, dx \quad \text{and} \quad |D(\mathcal{B}_\lambda u)|(\mathbb{R}^n) \leq c |Du|(\mathcal{O}_\lambda).$$

As a consequence, we have

$$(3.7) \quad \|T_\lambda u\|_q \leq c \|u\|_q \quad \text{and} \quad |D(T_\lambda u)|(\mathbb{R}^n) \leq c |Du|(\mathbb{R}^n).$$

Proof. Recall that $BV(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$. Since, $\text{supp}(\varphi_j * (\eta_j(u - u_j))) \subset Q_j$, we directly find by Young's convolution inequality:

$$\int_{\mathbb{R}^n} |\mathcal{B}_j u|^q \, dx = \int_{\frac{3}{4}Q_j} |\eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j))|^q \, dx \leq c \int_{Q_j} |u|^q \, dx.$$

Moreover, for each $j \in \mathbb{N}$ we obtain

$$\begin{aligned} |D(\mathcal{B}_j u)|(\mathbb{R}^n) &= |D(\eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j)))|(\mathbb{R}^n) \\ &\leq 2 |D(\eta_j(u - u_j))|(\mathbb{R}^n) \\ &\leq c |Du|(\tfrac{3}{4}Q_j) + c \int_{Q_j} \frac{|u - u_j|}{r_j} \, dx \\ &\leq c |Du|(Q_j), \end{aligned}$$

where we used Poincaré's inequality in the last step. This yields (3.5). Now, (3.6) follows by summing over j and using the finite intersection property of the Q_j , cf. (W5). Finally, (3.7) is a straightforward consequence of (3.4) and (3.6). The proof is complete. \square

We will now show that $T_\lambda u$ is in fact a Lipschitz continuous function.

Lemma 7. *There exists a constant $c = c(n) > 0$ such that for all $\lambda > 0$ there holds*

$$(3.8) \quad \mathcal{M}(DT_\lambda u) \leq c \frac{\lambda}{h(\lambda)^{n+1}}.$$

Proof. Let Q be an open cube with side length r . We use the alternatives of Lemma 3. We begin with alternative (A1). In this case, there exists $k \in \mathbb{N}$ such that $Q \cap \frac{1}{2}Q_k \neq \emptyset$, $8r \leq r_k$ and $Q \subset \frac{3}{4}Q_k$. Then $T_\lambda u = \sum_{j \in A_k} (\eta_j u_j + \varphi_j * (\eta_j(u - u_j)))$ on Q by (W2) and therefore

$$\begin{aligned} DT_\lambda u &= \sum_{j \in A_k} \nabla \left(\eta_j(u_j - u_k) + \varphi_j * (\eta_j(u - u_j)) \right) \\ &= \sum_{j \in A_k} \left(\nabla \eta_j(u_j - u_k) \right) + \sum_{j \in A_k} \nabla \varphi_j * (\eta_j(u - u_j)). \end{aligned}$$

Thus, by (P2) and $\mathcal{M}_Q f \leq |f|$ for all $f \in L^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{M}_Q(DT_\lambda u) &\leq \sum_{j \in A_k} \mathcal{M}_Q((u_j - u_k) \nabla \eta_j) + \sum_{j \in A_k} \mathcal{M}_Q(\nabla \varphi_j * (\eta_j(u - u_j))) \\ &\leq \sum_{j \in A_k} \mathcal{M}_Q((u_j - u_k) \nabla \eta_j) + \sum_{j \in A_k} \|\nabla \varphi_j * (\eta_j(u - u_j))\|_\infty \\ &\leq c \sum_{j \in A_k} \frac{|u_j - u_k|}{r_k} + \sum_{j \in A_k} \|\nabla \varphi_j\|_\infty \int_{Q_j} |u - u_j| dx \\ &\leq c \sum_{j \in A_k} \frac{|u_j - u_k|}{r_k} + c \sum_{j \in A_k} \varepsilon_j^{-n-1} \int_{Q_j} |u - u_j| dx \end{aligned}$$

using that $\text{supp}(\eta_j) \subset \frac{3}{4}Q_j$, $\varepsilon_j \leq \frac{1}{4}r_j$, and the properties of φ_j . Now, Lemma 4, $\varepsilon_j = h(\lambda) \frac{1}{4}r_j$ and $h(\lambda) \leq 1$ imply

$$(3.9) \quad \mathcal{M}_Q(DT_\lambda u) \leq c \lambda h(\lambda)^{-n-1}.$$

We turn to alternative (A2). In particular, for all $j \in \mathbb{N}$ with $Q \cap \frac{3}{4}Q_j \neq \emptyset$, there holds $r_j \leq 16r$ and $|Q_j| \leq 8^n |Q_j \cap Q|$. Moreover, $137Q \cap (\mathbb{R}^n \setminus \mathcal{O}_\lambda) \neq \emptyset$. Recall that $T_\lambda u = u - \sum_j \mathcal{B}_j u$ with convergence of the sum in the norm topology on $BV(\mathbb{R}^n)$, see Lemma 6. Thus,

$$(3.10) \quad \mathcal{M}_Q(DT_\lambda u) \leq \mathcal{M}_Q(Du) + \sum_{j: Q \cap \frac{3}{4}Q_j \neq \emptyset} \mathcal{M}_Q(D\mathcal{B}_j u).$$

We address the estimation of the single terms $\mathcal{M}_Q(D\mathcal{B}_j u)$ first. We start by noting that for any $v \in W^{1,1}(\mathbb{R}^n)$ with support in $\frac{3}{4}Q_j$ there holds

$$\begin{aligned} \mathcal{M}_{Q_j}(\varphi_j * Dv) &\leq \frac{1}{|Q_j|} \int_{Q_j} \int_{\mathbb{R}^n} |\varphi_j(y)| |Dv(x - y)| dy dx \\ &= \frac{1}{|Q_j|} \int_{\mathbb{R}^n} |\varphi_j(y)| \int_{Q_j} |Dv(x - y)| dx dy \\ &\leq \int_{Q_j} |Dv| dx = \mathcal{M}_{Q_j}(Dv) \end{aligned}$$

since $\text{supp}(\eta_j) + \text{supp}(\varphi_j) \subseteq Q_j$. If $v \in \text{BV}(\mathbb{R}^n)$ has support in $\frac{3}{4}Q_j$, choose a sequence $(v_k) \subset W^{1,1}(\mathbb{R}^n)$ such that $\text{supp}(v_k) \subset \frac{3}{4}Q_j$ and $v_k \rightarrow v$ strictly in $\text{BV}(\mathbb{R}^n)$. Clearly, $\varphi_j * v_k \rightarrow \varphi_j * v$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ and since $D(\varphi_j * v) = \varphi_j * Dv$, lower semicontinuity of the total variation with respect to L^1_{loc} -convergence implies

$$\begin{aligned} \mathcal{M}_{Q_j}(\varphi_j * Dv) &\leq \mathcal{M}_{Q_j}(D(\varphi_j * v)) \\ &\leq \liminf_{k \rightarrow \infty} \mathcal{M}_{Q_j}(D(\varphi_j * v_k)) \\ &= \liminf_{k \rightarrow \infty} \mathcal{M}_{Q_j}(\varphi_j * Dv_k) \leq \liminf_{k \rightarrow \infty} \mathcal{M}_{Q_j}(Dv_k) \leq \mathcal{M}_{Q_j}(Dv) \end{aligned}$$

provided $\text{supp}(Dv)$ is a closed subset of Q_j . Applying the previous inequality to $v = \eta_j(u - u_j)$, we estimate

$$\begin{aligned} \mathcal{M}_Q(D\mathcal{B}_j u) &\leq \mathcal{M}_Q(D(\eta_j(u - u_j))) + \mathcal{M}_Q(\varphi_j * D(\eta_j(u - u_j))) \\ (3.11) \quad &\leq \left(\frac{|Q_j|}{|Q|} \left(\mathcal{M}_{Q_j}(D(\eta_j(u - u_j))) + \mathcal{M}_{Q_j}(\varphi_j * D(\eta_j(u - u_j))) \right) \right) \\ &\leq c \left(\frac{|Q_j \cap Q|}{|Q|} \left(\mathcal{M}_{Q_j}(D(\eta_j(u - u_j))) \right) \right), \end{aligned}$$

the geometric alternative (A2) having entered in the last step only. By Lemma 4(a), we thus obtain

$$(3.12) \quad \mathcal{M}_{Q_j}(D(\eta_j(u - u_j))) \leq c \int_{Q_j} \frac{|u - u_j|}{r_j} dx + c \mathcal{M}_{Q_j}(Du) \leq c \lambda.$$

Thus, combining (3.10), (3.11) and (3.12), (A2) and the finite intersection of the Q_j 's, cf. (W5), imply

$$\mathcal{M}_Q(\nabla T_\lambda u) \leq \mathcal{M}_{137Q}(Du) + c \sum_{j: Q \cap \frac{3}{4}Q_j \neq \emptyset} \frac{|Q_j \cap Q|}{|Q|} \lambda \leq c \lambda.$$

Recalling (3.9) and that $h: (0, \infty) \rightarrow (0, 1]$, the proof is hereby complete. \square

The following corollary justifies the name *Lipschitz truncation*.

Corollary 8. *For each $\lambda > 0$ we have $T_\lambda u \in W^{1,\infty}(\mathbb{R}^n)$. More precisely, there exists $c = c(n) > 0$ such that for all $\lambda > 0$ there holds $\|\nabla T_\lambda u\|_\infty \leq c \frac{\lambda}{h(\lambda)^{n+1}}$.*

Proof. This is a direct consequence of Lemma 7 and Lemma 2(c). \square

We now turn to the convergence properties of $T_\lambda u \rightarrow u$ as $\lambda \rightarrow \infty$. The core feature of our truncation operator T_λ is that it possesses a nice (almost) dual operator S_λ which satisfies $DT_\lambda \approx S_\lambda^* D$, see (3.14). Let us define for $\rho \in C_c(\Omega; \mathbb{R}^n)$

$$(3.13) \quad S_\lambda \rho := \rho - \sum_j \eta_j(\rho - \varphi_j * \rho) = \rho \mathbb{1}_{\mathcal{O}_\lambda^c} + \sum_j \eta_j(\varphi_j * \rho).$$

Lemma 9 (Commutator type estimate). *The operator S_λ as given in (3.13) satisfies the following:*

- (a) S_λ is non-expansive for the L^∞ -norm in the sense that for all $\rho \in C_c(\Omega; \mathbb{R}^n)$ there holds

$$\|S_\lambda \rho\|_\infty \leq \|\rho\|_\infty,$$

- (b) For all $\rho \in C_c(\Omega; \mathbb{R}^n)$ and $u \in \text{BV}(\mathbb{R}^n)$ we have the commutator-type estimate

$$(3.14) \quad |\langle DT_\lambda u, \rho \rangle - \langle Du, S_\lambda \rho \rangle| \leq c h(\lambda) |Du|(\mathcal{O}_\lambda) \|\rho\|_\infty.$$

Proof. The claim of (a) follows by the pointwise estimate

$$|S\rho| \leq |\rho| \mathbb{1}_{\mathcal{O}_\lambda^c} + \sum_j \eta_j |\varphi_j * \rho| \leq \|\rho\|_\infty \mathbb{1}_{\mathcal{O}_\lambda^c} + \sum_j \eta_j \|\rho\|_\infty \leq \|\rho\|_\infty.$$

Let us turn to the proof of (b). By a routine approximation argument, it suffices to consider $\rho \in C_c^1(\Omega; \mathbb{R}^n)$ with $\|\rho\|_\infty \leq 1$. Then

$$\begin{aligned} & \langle DT_\lambda u, \rho \rangle - \langle Du, S_\lambda \rho \rangle \\ &= -\langle T_\lambda u, \operatorname{div}(\rho) \rangle - \langle Du, S_\lambda \rho \rangle \\ &= -\left\langle u - \sum_j \left(\eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j)) \right), \operatorname{div} \rho \right\rangle \\ &\quad - \langle Du, \rho - \sum_j \eta_j(\rho - \varphi_j * \rho) \rangle \\ (3.15) \quad &= -\left\langle \sum_j D \left(\eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j)) \right), \rho \right\rangle + \langle Du, \sum_j \eta_j(\rho - \varphi_j * \rho) \rangle \\ &= -\left\langle \sum_j \left(\eta_j Du - \varphi_j * (\eta_j Du) \right), \rho \right\rangle + \langle Du, \sum_j \eta_j(\rho - \varphi_j * \rho) \rangle \\ &\quad - \left\langle \sum_j \left(\nabla \eta_j(u - u_j) - \varphi_j * (\nabla \eta_j(u - u_j)) \right), \rho \right\rangle \\ &= -\left\langle \sum_j \left(\nabla \eta_j(u - u_j) - \varphi_j * (\nabla \eta_j(u - u_j)) \right), \rho \right\rangle. \end{aligned}$$

In particular,

$$|\langle DT_\lambda u, \rho \rangle - \langle Du, S_\lambda \rho \rangle| \leq \sum_j \int_{\frac{3}{4}Q_j} |((u - u_j) \nabla \eta_j) - \varphi_j * ((u - u_j) \nabla \eta_j)| \, dx.$$

Now, we use the well known mollifier estimate

$$(3.16) \quad \|v - \varphi_j * v\|_1 \leq c \varepsilon_j |Dv|(\mathbb{R}^n).$$

Indeed, the $W^{1,1}$ -case can be found in [MZ97], while the BV case follows by approximation in the strict topology. Hence,

$$\begin{aligned} |\langle DT_\lambda u, \rho \rangle - \langle Du, S_\lambda \rho \rangle| &\stackrel{(3.16)}{\lesssim} \sum_j \varepsilon_j |D(\nabla \eta_j(u - u_j))|(Q_j) \\ &\lesssim \sum_j \frac{\varepsilon_j}{r_j} \left(\int_{Q_j^*} \frac{|u - u_j|}{r_j} \, dx + |Du|(Q_j) \right) \\ &\leq h(\lambda) \sum_j |Du|(Q_j) \\ &\leq h(\lambda) |Du|(\mathcal{O}_\lambda). \end{aligned}$$

This is (b), and the proof is complete. \square

We are now able to characterize to prove area-strict convergence.

Lemma 10 (Area-strict convergence). *We have $T_\lambda u \rightarrow u$ in the area-strict sense of $BV(\mathbb{R}^n)$ as $\lambda \rightarrow \infty$. In particular, $DT_\lambda u \rightarrow Du$ area strictly for $\lambda \rightarrow \infty$. Moreover,*

$$(3.17) \quad \langle DT_\lambda u \rangle(\mathbb{R}^n) \leq \langle Du \rangle(\mathbb{R}^n) + ch(\lambda) |Du|(\mathcal{O}_\lambda) + \frac{c}{\lambda} |Du|(\mathbb{R}^n),$$

$$(3.18) \quad |DT_\lambda u|(\mathbb{R}^n) \leq |Du|(\mathbb{R}^n) + ch(\lambda) |Du|(\mathcal{O}_\lambda).$$

Proof. We start with the L^1 convergence. Lemma 6, $BV(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n)$ and $|\mathcal{O}_\lambda| \leq \frac{c}{\lambda} |Du|(\mathbb{R}^n)$ (which follows from Lemma 2 (b)) imply

$$\|u - T_\lambda u\|_1 = \|\mathcal{B}_\lambda u\|_1 \leq c \int_{\mathcal{O}_\lambda} |u| dx \leq c \|u\|_{\frac{n}{n-1}} \left(\frac{|Du|(\mathbb{R}^n)}{\lambda} \right)^{\frac{1}{n}} \rightarrow 0, \quad \lambda \rightarrow \infty.$$

Next, recall that the area-strict convergence of $DT_\lambda u$ to Du is equivalent to strict convergence of $(DT_\lambda u, \mathcal{L}^n)$ to (Du, \mathcal{L}^n) . To prove the latter, let $\rho_1 \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\rho_2 \in C_c^1(\mathbb{R}^n)$ be such that $\sqrt{|\rho_1|^2 + |\rho_2|^2} \leq 1$. We estimate

$$\begin{aligned} |\langle (DT_\lambda u, \mathcal{L}^n), (\rho_1, \rho_2) \rangle| &= |\langle DT_\lambda u, \rho_1 \rangle + \langle \mathcal{L}^n, \rho_2 \rangle| \\ &= |\langle Du, S_\lambda \rho_1 \rangle + (\langle DT_\lambda u, \rho_1 \rangle - \langle Du, S_\lambda \rho_1 \rangle) + \langle \mathcal{L}^n, \rho_2 \rangle| \\ &= |\langle (Du, \mathcal{L}^n), (S_\lambda \rho_1, S_\lambda \rho_2) \rangle + (\langle DT_\lambda u, \rho_1 \rangle - \langle Du, S_\lambda \rho_1 \rangle) + \langle \mathcal{L}^n, \rho_2 - S_\lambda \rho_2 \rangle| \\ &\leq |\langle (Du, \mathcal{L}^n), (S_\lambda \rho_1, S_\lambda \rho_2) \rangle| + |(\langle DT_\lambda u, \rho_1 \rangle - \langle Du, S_\lambda \rho_1 \rangle)| + |\langle \mathcal{L}^n, \rho_2 - S_\lambda \rho_2 \rangle| \\ &=: \text{I} + \text{II} + \text{III}. \end{aligned}$$

By Lemma 9(a), S_λ is non-expansive for the L^∞ -norm and thus

$$|(S_\lambda \rho_1, S_\lambda \rho_2)| \leq \sqrt{\|\rho_1\|_\infty^2 + \|\rho_2\|_\infty^2} \leq 1.$$

Hence, $\text{I} \leq \langle Du \rangle(\mathbb{R}^n)$. For II , we utilise Lemma 9(b) to find

$$\text{II} \leq ch(\lambda) |Du|(\mathcal{O}_\lambda) \|\rho_1\|_\infty \leq ch(\lambda) |Du|(\mathcal{O}_\lambda) \xrightarrow{\lambda \rightarrow \infty} 0$$

using $h(\lambda) \rightarrow 0$ and $|Du|(\mathcal{O}_\lambda) \leq |Du|(\mathbb{R}^n) < \infty$. Ad III . Using $\rho_2 = S_\lambda \rho_2 \leq 1$ on \mathcal{O}_λ^c , $\|S_\lambda \rho_2\|_\infty \leq \|\rho_2\|_\infty$ and $|\mathcal{O}_\lambda| \leq \frac{c}{\lambda} |Du|(\mathbb{R}^n)$,

$$|\langle \mathcal{L}^n, \rho_2 - S_\lambda \rho_2 \rangle| \leq 2 \|\rho_2\|_\infty |\mathcal{O}_\lambda| \leq \frac{c}{\lambda} |Du|(\mathbb{R}^n) \xrightarrow{\lambda \rightarrow \infty} 0.$$

In consequence, gathering the estimates for $\text{I}, \text{II}, \text{III}$,

$$(3.19) \quad \langle DT_\lambda u \rangle(\mathbb{R}^n) \leq \langle Du \rangle(\mathbb{R}^n) + ch(\lambda) |Du|(\mathcal{O}_\lambda) + \frac{c}{\lambda} |Du|(\mathbb{R}^n).$$

This proves, (3.17). The estimate (3.18) follows analogously without the use of ρ_2 . Hence,

$$(3.20) \quad \limsup_{\lambda \rightarrow \infty} |\langle DT_\lambda u \rangle(\mathbb{R}^n)| \leq \langle Du \rangle(\mathbb{R}^n).$$

On the other hand, by the first part of the proof, $T_\lambda u \rightarrow u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$. Thus, by the L^1 -lower semicontinuity (2.3) we obtain

$$\langle Du \rangle(\mathbb{R}^n) \leq \liminf_{\lambda \rightarrow \infty} \langle DT_\lambda u \rangle(\mathbb{R}^n).$$

In conjunction with (3.20), this yields $\lim_{\lambda \rightarrow \infty} \langle DT_\lambda u \rangle(\mathbb{R}^n) = \langle Du \rangle(\mathbb{R}^n)$ and the proof is complete. \square

We conclude by identifying the limits of the single constituents of $T_\lambda u$:

Lemma 11. *The following hold:*

- (a) $\mathbb{1}_{\mathcal{O}_\lambda^c} \nabla T_\lambda u = \mathbb{1}_{\mathcal{O}_\lambda^c} \nabla u \rightarrow \nabla u$ in $L^1(\mathbb{R}^n)$ as $\lambda \rightarrow \infty$.
- (b) $\nabla T_\lambda \mathcal{L}^n \llcorner \mathcal{O}_\lambda \rightarrow D^s u$ in the sense of area-strict convergence of \mathbb{R}^n -valued Radon measures.

Proof. Since $|\mathcal{O}_\lambda| \rightarrow 0$ as $\lambda \rightarrow \infty$ and the approximate gradient satisfies $\nabla u \in L^1(\mathbb{R}^n)$, we have $\nabla u - \mathbb{1}_{\mathcal{O}_\lambda^c} \nabla u = \mathbb{1}_{\mathcal{O}_\lambda} \nabla u \rightarrow 0$ in $L^1(\mathbb{R}^n)$. This proves (a). Ad (b). Let $\varphi \in C_c(\mathbb{R}^n)$. By Lemma 10 it follows that $DT_\lambda u \rightarrow Du$ in the weak* sense, so

$$\langle \mathbb{1}_{\mathcal{O}_\lambda^c} \nabla T_\lambda u, \varphi \rangle = \langle \nabla T_\lambda u, \varphi \rangle - \langle \mathbb{1}_{\mathcal{O}_\lambda} \nabla T_\lambda u, \varphi \rangle \rightarrow \langle Du - \nabla u \mathcal{L}^n, \varphi \rangle = \langle D^s u, \varphi \rangle.$$

It thus remains to establish that $\langle DT_\lambda u \rangle(\mathcal{O}_\lambda) \rightarrow \langle D^s u \rangle(\mathbb{R}^n)$ as $\lambda \rightarrow \infty$. To this end, we record that

$$\begin{aligned} \langle DT_\lambda u \rangle(\mathcal{O}_\lambda) &= \langle DT_\lambda u \rangle(\mathbb{R}^n) - \langle DT_\lambda u \rangle(\mathcal{O}_\lambda^c) \\ &= \langle DT_\lambda u \rangle(\mathbb{R}^n) - \langle Du \rangle(\mathcal{O}_\lambda^c) \\ &= \langle DT_\lambda u \rangle(\mathbb{R}^n) - (\langle \nabla u \mathcal{L}^n \rangle(\mathbb{R}^n) - \langle \nabla u \mathcal{L}^n \rangle(\mathcal{O}_\lambda)) \\ &\leq (\langle DT_\lambda u \rangle(\mathbb{R}^n) - \langle \nabla u \mathcal{L}^n \rangle(\mathbb{R}^n)) + |\mathcal{O}_\lambda| + |\nabla u \mathcal{L}^n|(\mathcal{O}_\lambda) \\ &\rightarrow \langle Du \rangle(\mathbb{R}^n) - \langle \nabla u \rangle(\mathbb{R}^n) = \langle D^s u \rangle(\mathbb{R}^n), \quad \lambda \rightarrow \infty, \end{aligned}$$

where we have used that $\nabla u \mathcal{L}^n \llcorner \mathcal{O}_\lambda^c = Du \llcorner \mathcal{O}_\lambda^c$ in the third equality, the trivial bound $\sqrt{1 + |\cdot|^2} \leq 1 + |\cdot|$ in the fourth and $|\mathcal{O}_\lambda| \rightarrow 0$ in conjunction with (a) in the ultimate line. This establishes $\limsup_{\lambda \rightarrow \infty} \langle DT_\lambda u \rangle(\mathcal{O}_\lambda) \leq \langle D^s u \rangle(\mathbb{R}^n)$. On the other hand, the L^1 lower semicontinuity (2.3) implies $\langle Du \rangle(\mathbb{R}^n) \leq \liminf_{\lambda \rightarrow \infty} \langle DT_\lambda u \rangle(\mathbb{R}^n)$ so that, in total, $\langle Du \rangle(\mathbb{R}^n) = \lim_{\lambda \rightarrow \infty} \langle DT_\lambda u \rangle(\mathbb{R}^n)$. The proof is complete. \square

3.4. Preserving zero boundary values. Sometimes it is desirable to preserve zero boundary values of a given function. We show in this section how to modify our Lipschitz truncation such that the u_λ also have zero boundary values.

Hence, let Ω be a bounded Lipschitz domain and let $u \in \text{BV}(\mathbb{R}^n)$ with $u = 0$ on $\mathbb{R}^n \setminus \Omega$. We take the same decomposition of our bad set by a Whitney cover as in the beginning of the section. Recall that

$$\begin{aligned} \mathcal{B}_j u &= \eta_j(u - u_j) - \varphi_j * (\eta_j(u - u_j)), \\ T_\lambda u &= u_\lambda = u - \sum_j \mathcal{B}_j u. \end{aligned}$$

To obtain $T_\lambda u = 0$ on $\mathbb{R}^n \setminus \Omega$, we have to ensure that $\mathcal{B}_j u = 0$ on $\mathbb{R}^n \setminus \Omega$. For this, let Q_j be a cube close to the boundary $\partial\Omega$, i.e. $\frac{3}{4}Q_j \not\subset \Omega$. In this case the definition of the u_j in (3.1) ensures that $u_j = 0$. Thus, in this case

$$\mathcal{B}_j u = \eta_j u - \varphi_j * (\eta_j u).$$

By assumption on u , we have $\eta_j(u - u_j) = \eta_j u = 0$ on $\mathbb{R}^n \setminus \Omega$. However, the convolution with φ_j might transport values of u to $\mathbb{R}^n \setminus \Omega$. To avoid this, it is necessary to use a directed convolution. So have to drop the assumption that the φ_j are radially symmetric mollifiers.

By Lemma 2 (a), we have

$$|Q_j| \leq \mathcal{L}^n(\{\mathcal{M}(Du) > \lambda\}) \lesssim \frac{|Du|(\mathbb{R}^n)}{\lambda}.$$

Thus, for large λ the Whitney cubes are small. Now, since Ω is a Lipschitz domain, its boundary $\partial\Omega$ can be written locally on Q_j as a graph of a Lipschitz function. Thus, there exists a unit vector ν_j (an approximation of the normal of $\partial\Omega$ on Q_j) such that $Q_j \cap \Omega$ satisfies the outer cone condition in direction ν_j . Thus, we can choose $K = K(\Omega) \geq 1$ such that for all $x \in \Omega$ we have

$$(3.21) \quad \Omega^c + B_{\frac{1}{K}}\left(\frac{1}{2}\nu_j\right) = \left\{x + y : x \in \Omega^c, y \in B_{\frac{1}{K}}\left(\frac{1}{2}\nu_j\right)\right\} \subset \Omega^c.$$

Now, let φ be a smooth, non-negative, radially symmetric mollifier with support in the unit ball. Then we define the local directed mollifier φ_j by

$$\varphi_j(x) := (K\varepsilon_j)^{-n} \varphi\left(\frac{x}{K\varepsilon_j} + \frac{\nu_j}{2}\right) \quad \text{with} \quad \varepsilon_j := h(\lambda) \frac{1}{4} r_j.$$

Then (3.21) ensures that $\varphi_j * (\eta_j u) = 0$ on $\mathbb{R}^n \setminus \Omega$. The same holds for the $\mathcal{B}_j u$. Consequently, $u_\lambda = 0$ on $\mathbb{R}^n \setminus \Omega$ provided that λ is large enough depending on the boundary $\partial\Omega$. Note that since the φ_j are no longer radially symmetric, one

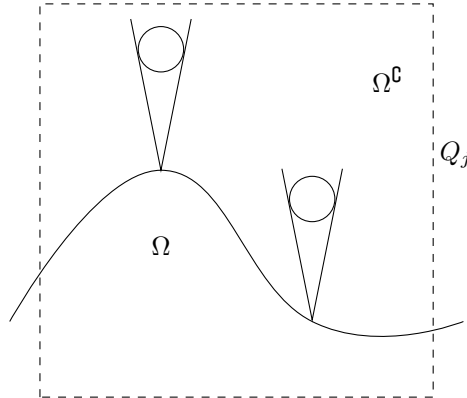


FIGURE 2. Local chart of the boundary. The cones indicate the direction of the convolution.

has to replace φ_j in the definition of the (almost) dual operator S_λ by $\overline{\varphi_j}$ with $\overline{\varphi_j}(x) := \varphi_j(-x)$.

3.5. Proof of Theorem 1. We are now in position to prove our main theorem.

For a given $\lambda > 0$, define $u_\lambda := T_\lambda u$ as in (3.4). The Lipschitz property (a) follows from Corollary 8. The smallness of the set $\{u \neq u_\lambda\}$ from (b) is an immediate consequence of the construction of $T_\lambda u$ and Lemma 2 (b). The stability assertions of (c) are given in Lemma 6. On the other hand, the convergence properties (d) follow from Lemma 10 and Lemma 11. The preservation of the zero boundary values (e) follows from Subsection 3.4. The proof of Theorem 1 is complete. \square

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