Functional Analysis & PDEs

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Dr. F. Gmeineder

Functional Analysis Revision

Problem 1: Warm-up on compact operators

Let $(X, \|\cdot\|)$ be a Banach space. Denote $K(X)$ the compact linear operators $T\colon X \to X$.

(a) Prove that $K(X)$ is a closed, two-sided ideal in $\mathcal{L}(X)$ for the operator norm on X. *In particular, the quotient $\mathcal{C}(X) := \mathcal{L}(X)/K(X)$ is well-defined; it is called the Calkin-Algebra of X.*

(b) Prove that $K(X)$ is dense in $\mathcal{L}(X)$ for the operator norm on X if and only if $\dim(X) < \infty$.

(c) Let $T\colon X \to X$ be a linear operator such that $\|x\| \leq c\|Tx\|$ holds for some $c > 0$ and all $x \in X$. Prove that T is compact if and only if $\dim(X) < \infty$.

(d) Let $k \in \mathrm{C}([0,1] \times [0,1])$ and define $T\colon\ \mathrm{C}([0,1]) \to \mathrm{C}([0,1])$ by

$$Tf(x) := \int_{[0,1]} k(x,y)f(y)\,\mathrm{d}y, \qquad x \in [0,1].$$

Prove that T is compact.

Problem 2: True or false?

Decide whether the following statements are true in general – justify your answer.

(a) Any separable real Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$.

(b) Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose that $X \subset \mathrm{W}^{1,p}(\mathrm{B}_1(0))$ is such that for each $u \in X$ the trivial extension belongs to $\mathrm{W}^{1,p}(\mathbb{R}^n)$. Then $\mathrm{W}^{1,p}(\Omega)$ is compactly embedded into $\mathrm{L}^1(\Omega)$.

(c) There exists a functional $f \in (\ell^\infty)^*$ such that $f \neq 0$ and $f(x) = 0$ for all $x \in X$, where $X := \bigcup_{1 \leq p < \infty} \ell^p$.

Problem 3:

Let $(X, \|\cdot\|)$ be a real Banach space. Moreover, let $f, f_1, f_2, ... \in X^*$ and $x, x_1, x_2, ... \in X$ be such that

$$f_j \overset{*}{\rightharpoonup} f \quad \text{and} \quad x_j \to x \quad \text{as } j \to \infty.$$

Prove that $f_j(x_j) \to f(x)$ as $j \to \infty$.

Problem 4:

Let $a \in \mathbb{R}$ and define $u_a \colon \mathbb{R} \to \mathbb{R}$ by

$$u_a(x) := \begin{cases} ae^x & x < \infty, \\ 10 & x = 0, \\ e^{-x}\cos(x) & x > 0. \end{cases}$$

Determine the maximal set of a's such that $u_a \in \mathrm{W}^{1,1}(\mathbb{R})$.