Functional Analysis & PDEs

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Functional Analysis Revision

Problem 1: **10 marks**

Prove or disprove whether

$$X := \{x = (x_j) \in \ell^1(\mathbb{N}) \colon \sum_j x_j = 0\}$$

is dense in $(\ell^2(\mathbb{N}), \|\cdot\|_{\ell^2(\mathbb{N})})$.

Solution. Since $\ell^1 \subset \ell^2$ (also see the next problems), X is obviously a linear subspace of ℓ^2. Suppose that X is not dense in $\mathcal{H}$. Then $\overline{X} \subsetneq \mathcal{H}$, and we find $z \in \mathcal{H} \setminus \overline{X}$. Since $\overline{X}$ is closed, we may consider its orthogonal projection $\Pi(z)$ onto $\overline{X}$. Put $x := z - \Pi(z)$. Then $\langle x, y\rangle = \langle z, y\rangle - \langle \Pi(z), y\rangle = 0$ for all $y \in \overline{X}$. Clearly $x \neq 0$ as otherwise $z = \Pi(z)$ and so $z \in \overline{X}$. Therefore, there exists a non-zero $x = (x_j) \in \ell^2$ such that $\langle x, y\rangle = 0$ for all $y \in X$. Note that, for all $i \neq j$, $e_i - e_j \in X$, where $e_i = (\delta_{ik})_k$. Therefore, $\langle x, e_i - e_j\rangle = 0$. This implies

$$x_i = x_j \qquad \text{for all } i \neq j,$$

and since $x \neq 0$, $x_i = c \neq 0$. But then $x \notin \ell^2$, a contradiction. ∎

Problem 2: **4 + 6 = 10 marks**

Let $1 \leq p \leq q \leq \infty$.

(a) Prove that $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.

(b) Let $T\colon \ell^p(\mathbb{N}) \ni x \mapsto x \in \ell^q(\mathbb{N})$ be the injection underlying (a). Show that T is a bounded linear operator and compute its operator norm.

Solution. We prove both assertions simultaneously. Let $\xi = (\xi_j) \in \ell^p \setminus \{0\}$ and put $x := \xi/\|\xi\|_{\ell^p}$. Then, for any $j \in \mathbb{N}$,

$$|\xi_j| \leq \Big(\sum_i |\xi_i|^p \Big)^{\frac{1}{p}} \Rightarrow \forall j \in \mathbb{N}\colon\ |x_j| \leq 1.$$

Hence $|x_j|^q \leq |x_j|^p$ for $q \geq p$. Thus, we have

$$\sum_j |x_j|^q \leq \sum_j |x_j|^p = 1.$$

Therefore,

$$\frac{\|\xi\|_{\ell^q}}{\|\xi\|_{\ell^p}} = \|x\|_{\ell^q} \leq 1.$$

Note that if $\xi = 0$, then there is nothing to prove. Hence $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$, and the identity $\mathrm{Id}\colon \ell^p(\mathbb{N}) \hookrightarrow \ell^q(\mathbb{N})$ has operator norm at most one. One the other hand, consider $e_1 = (1, 0, 0, ...) \in \ell^p(\mathbb{N})$. Then $\mathrm{Id}(e_1) = e_1$, and $\|e_1\|_{\ell^q(\mathbb{N})} = 1$. Hence $\|\mathrm{Id}\|_{\ell^p \to \ell^q} = 1$. ∎

Problem 3: **10 marks**

Let $1 \leq q \leq \infty$. Prove that

$$\bigcup_{p<q} \ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N}).$$

Solution. By problem 2, we only have to establish that the inclusions are strict provided $1 \leq p < q \leq \infty$. If $q = \infty$, pick any $x = (x_j)$ which is bounded but does not converge to zero – then $x \notin \bigcup_{p<\infty} \ell^p(\mathbb{N})$. Now let $q < \infty$. Suppose that $\ell^q(\mathbb{N}) = \bigcup_{1 \leq p < q} \ell^p(\mathbb{N})$. We pick a sequence $(p_k) \subset [1, \infty)$ with $p_k \nearrow q$. Then

$$\bigcup_{p<q} \ell^p(\mathbb{N}) = \bigcup_{k \in \mathbb{N}} \ell^{p_k}(\mathbb{N}).$$

Indeed, if x belongs to the left-hand side, then there exists $p < q$ with $x \in \ell^p(\mathbb{N})$. But $p_k \nearrow q$, so that there exists $k_0 \in \mathbb{N}$ with $p < p_{k_0}$ and hence $x \in \ell^{p_{k_0}}(\mathbb{N})$, and hence x belongs to the right-hand side, too. The other inclusion is trivial. Now put, for $N \in \mathbb{N}$,

$$\ell^{p_k}_N(\mathbb{N}) := \{x \in \ell^{p_k}(\mathbb{N})\colon\ \|x\|_{\ell^{p_k}} \leq N\}. \tag{3.1}$$

First, $\ell^{p_k}_N(\mathbb{N})$ is closed (for $\|\cdot\|_{\ell^q}$). Indeed, let $x, x^1, x^2, ... \in \ell^{p_k}_N(\mathbb{N})$ such that $x^j \to x$ in $\ell^q(\mathbb{N})$. Then, by Fatou's lemma (on $\mathbb{N}$ with the counting measure), $x^j_i \to x_i$ for all $i \in \mathbb{N}$. Therefore,

$$\|x\|_{\ell^{p_k}} \leq \liminf_{j\to\infty} \|x^j\|_{\ell^{p_k}} \leq N.$$

Now, $\ell^{p_k}_N(\mathbb{N})$ is closed in $\ell^{p_k}(\mathbb{N})$ and, as a proper closed subset of $\ell^q(\mathbb{N})$, has empty interior. Note that $\ell^{p_k} \subsetneq \ell^q$ – indeed, pick $x = (\frac{1}{j^{\frac{1}{p_k}}})$. In conclusion,

$$\ell^q(\mathbb{N}) = \bigcup_{k=1}^{\infty} \bigcup_{N \in \mathbb{N}} \ell^{p_k}_N(\mathbb{N})$$

represents $\ell^q(\mathbb{N})$ as the countable union of nowhere dense sets (with respect to the $\ell^q(\mathbb{N})$-norm) – a contradiction to Baire's theorem since $\ell^q(\mathbb{N})$ is Banach for $\|\cdot\|_{\ell^q(\mathbb{N})}$. The proof is complete. ∎

Problem 4: **10 marks**

Let $\mathrm{C}([0,1])$ the space of continuous functions $u\colon [0,1] \to \mathbb{R}$, endowed with the usual supremum norm. Let $X \subset \mathrm{C}([0,1])$ be a closed subspace of $\mathrm{C}([0,1])$ for the supremum norm which satisfies

$$X \subset \bigcup_{0<\alpha\leq 1} \mathrm{C}^{0,\alpha}([0,1]).$$

Prove that $\dim(X) < \infty$.

Solution. We aim to show that all bounded sequences in $(X, \|\cdot\|)$ possess a convergent subsequence in $(X, \|\cdot\|_{\sup})$. By the compactness characterisation of finite dimensional spaces, this shall establish the claim.

To this end, we employ the Arzelá-Ascoli theorem. The underlying base space $[0,1]$ is compact, ensuring the applicability of the latter theorem. Let $(f_i) \subset X$ be bounded. The Ascoli-Arzelá theorem requires *equicontinuity*, and so we must strive for equicontinuity first. We claim that there exist $\beta \in (0,1]$ and $C > 0$ such that for all $i \in \mathbb{N}$ and all $x, y \in [0,1]$ there holds

$$|f_i(x) - f_i(y)| \leq C|x-y|^{\beta}. \tag{4.1}$$

Now,

$$X = \bigcup_{\substack{n\in\mathbb{N}\\ j\in\mathbb{N}}} \{u \in X\colon\ [u]_{\mathrm{C}^{0,\frac{1}{n}}} \leq j\},$$

and the union on the right hand side over $n, j \in \mathbb{N}$ is countable. We now claim that each $X_{j,n} := \{u \in X\colon\ [u]_{\mathrm{C}^{0,\frac{1}{n}}} \leq j\}$ is closed in $(X, \|\cdot\|_{\sup})$. Let $\varphi_1, ... \in X_{j,n}$ and $\varphi \in \mathrm{C}([0,1])$ be such that $\varphi_k \to \varphi$ with respect to $\|\cdot\|_{\sup}$. Then there holds

$$\begin{aligned}|\varphi(x) - \varphi(y)| &\leq |\varphi(x) - \varphi_k(x)| + |\varphi_k(x) - \varphi_k(y)| + |\varphi_k(y) - \varphi(y)| \\ &\leq 2\|\varphi - \varphi_k\|_{\sup} + j|x-y|^{\frac{1}{n}} \\ &\overset{k\to\infty}{\longrightarrow} j|x-y|^{\frac{1}{n}},\end{aligned}$$

and hence $\varphi \in X_{j,n}$. Thus $X_{j,n}$ is closed in $(\mathrm{C}([0,1]), \|\cdot\|_{\sup})$. As a closed subspace of $\mathrm{C}([0,1])$, $(X, \|\cdot\|_{\sup})$ is Banach in its own right. Therefore, by Baire, there must exist $(n,j) \in \mathbb{N}\times\mathbb{N}$ such that $X_{j,n}$ has non-empty interior. This, in particular, means that there exists $\varphi_0 \in X_{j,n}$ and $\varepsilon > 0$ such that $\mathrm{B}(\varphi_0, \varepsilon) \subset X_{j,n}^{\circ}$. Now let $\varphi \in X \setminus \{0\}$. Then $\varphi_0 + \delta\varphi \in \mathrm{B}(\varphi_0, \varepsilon)$ for any $0 < \delta < \frac{\varepsilon}{\|\varphi\|_{\sup}}$. In consequence, we find for all $x, y \in [0,1]$:

$$\begin{aligned}|\varphi(x) - \varphi(y)| &= \frac{1}{\delta}|\delta\varphi(x) - \delta\varphi(y)| \\ &\leq \frac{1}{\delta}|\delta\varphi(x) + \varphi_0(x) - \delta\varphi(y) - \varphi_0(y)| + \frac{1}{\delta}|\varphi_0(x) - \varphi_0(y)| \\ &\leq \frac{2j}{\delta}|x-y|^{\frac{1}{n}}.\end{aligned}$$

Hence, sending $\delta \nearrow \frac{\varepsilon}{\|\varphi\|_{\sup}}$, we find

$$|\varphi(x) - \varphi(y)| \leq \frac{2j}{\varepsilon}\|\varphi\|_{\sup}|x-y|^{\frac{1}{n}}. \tag{4.2}$$

Coming back to the original task: Let $(f_i) \subset X$ be bounded for $\|\cdot\|_{\sup}$. Then $\sup_i \|f_i\|_{\sup} < \infty$. Estimate (4.2) entails that (4.1) is satisfied with the particular choice $C = \frac{2j}{\varepsilon}\sup_i \|f_i\|_{\sup} < \infty$. So (f_i) is equicontinuous and bounded, hence relatively compact in $\mathrm{C}([0,1])$, and since it is closed by assumption, there exists $f \in \mathrm{C}([0,1])$ such that $f_{i(j)} \to f$ in $\mathrm{C}([0,1])$ for a suitable subsequence. In conclusion, $\dim(X) < \infty$ and the proof is complete. ∎

Problem 5: **10 marks**

Let $\mathcal{H}$ be separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. Let (e_j) be an orthonormal basis for $\mathcal{H}$ and let (x_n) be a sequence in $\mathcal{H}$. Prove that the following are equivalent:

(a) For all $f \in \mathcal{H}^*$ there holds $f(x_n) \to 0$ as $n \to \infty$.

(b) For all $j \in \mathbb{N}$ there holds $\langle e_j, x_n\rangle \to 0$ as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.

Solution. Ad (a)$\Rightarrow$(b). Fix $j \in \mathbb{N}$. Then $\langle e_j, \cdot\rangle \in \mathcal{H}^*$ by Cauchy-Schwarz, and hence $\langle e_j, x_n\rangle \to 0$ by (a). For the second part, we employ the uniform boundedness principle. Define $\Phi_n \in \mathcal{H}^{**}$ via $\Phi_n\colon \mathcal{H}^* \ni f \mapsto f(x_n)$. Then, by the Riesz representation theorem, $\|\Phi_n\|_{\mathcal{H}^{**}} = \|x_n\|_{\mathcal{H}}$. The assumption from (a) implies that $\sup_{n\in\mathbb{N}} |\Phi_n(f)| < \infty$ for any $f \in \mathcal{H}^*$, and hence, by the uniform boundedness principle, $\sup_{n\in\mathbb{N}} \|\Phi_n\| < \infty$. Since $\|\Phi_n\| = \|x_n\|$, (b) follows at once.
Ad (b)$\Rightarrow$(a). Since $\mathcal{H}$ is separable, it possesses a countable orthonormal basis (e_j). By the Riesz representation theorem, any $f \in \mathcal{H}^*$ can be represented as $f(x) = \langle x, y\rangle$ for some $y \in \mathcal{H}$; without loss of generality, $y \neq 0$ as otherwise there is nothing to prove. Express $y = \sum_j \langle y, e_j\rangle e_j$. Now, for any $N \in \mathbb{N}$,

$$\begin{aligned} f(x_n) = \langle y, x_n\rangle &= \sum_j \langle y, e_j\rangle\langle e_j, x_n\rangle \\ &= \sum_{j\leq N} \langle y, e_j\rangle\langle e_j, x_n\rangle + \sum_{j\geq N} \langle y, e_j\rangle\langle e_j, x_n\rangle \\ &= \sum_{j\leq N} \langle y, e_j\rangle\langle e_j, x_n\rangle + \left\langle \sum_{j\geq N} \langle y, e_j\rangle e_j, x_n \right\rangle \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Then we find with $L := \sup_{n\in\mathbb{N}} \|x_n\|$ (which is finite by the second hypothesis of (b)),

$$\exists N_0 \in \mathbb{N}\colon \quad \|\sum_{j\geq N_0} \langle y, e_j\rangle e_j\| < \frac{\varepsilon}{2L}.$$

To make this precise, note that by Bessel's inequality,

$$\|\sum_{j\geq N_0} \langle y, e_j\rangle e_j\|^2 \leq \sum_{j\geq N_0} |\langle y, e_j\rangle|^2 \leq \|y\|^2 < \infty,$$

and so the existence of such a number N_0 follows (the mid series is absolutely summable). On the other hand, by the first hypothesis of (b), we find $n_1 \in \mathbb{N}$ such that for all $j \in \{1, ..., N_0\}$ and $n \geq n_1$ there holds

$$|\langle e_j, x_n\rangle| \leq \frac{1}{2N_0\|y\|}\varepsilon.$$

In conclusion, for all $n \geq n_1$ there holds

$$\begin{aligned} |f(x_n)| &\leq \Big(\sum_{1\leq j\leq N_0} |\langle y, e_j\rangle| \Big) \max_{1\leq j\leq N_0} |\langle e_j, x_n\rangle| + \frac{\varepsilon}{2} \\ &\leq \frac{N_0\|y\|}{2N_0\|y\|}\varepsilon + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This is (a), and the proof is complete. ∎