ON CRITICAL L^p–DIFFERENTIABILITY OF BD–MAPS

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ABSTRACT. We prove that functions of locally bounded deformation on $\mathbb{R}^n$ are $\mathrm{L}^{\frac{n}{n-1}}$–differentiable $\mathcal{L}^n$–almost everywhere. More generally, we show that this critical L^p–differentiability result holds for functions of locally bounded $\mathbb{A}$–variation, provided that the first order, homogeneous differential operator $\mathbb{A}$ has finite dimensional null–space.

1. INTRODUCTION

Approximate differentiability properties of weakly differentiable functions are reasonably well understood. Namely, it is well–known that maps in $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ are L^{p^*}–differentiable $\mathcal{L}^n$–a.e. in $\mathbb{R}^n$, where $1\leqslant p<n$, $p^*:=np/(n-p)$ (see, e.g., [5, Thm 6.2]). We recall that a map $u\colon\mathbb{R}^n\to\mathbb{R}^N$ is L^q–approximately differentiable at $x\in\mathbb{R}^n$ if and only if there exists a matrix $M\in\mathbb{R}^{N\times n}$ such that

$$\left(\fint_{\mathrm{B}_r(x)}|u(y)-u(x)-M(y-x)|^q\,\mathrm{d}y\right)^{\frac{1}{q}}=o(r)$$

as $r\downarrow 0$, whence, in particular, u is approximately differentiable at x with approximate gradient M (see Section 2 for precise definitions). For $p=1$ one can show in addition that maps $u\in\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n,\mathbb{R}^N)$ are L^{1^*}–differentiable $\mathcal{L}^n$–a.e. with the approximate gradient equal $\mathcal{L}^n$–a.e. to the absolutely continuous part of $\mathrm{D}u$ ([5, Thm. 6.1, 6.4]). It is natural to ask a similar question of the space $\mathrm{BD}(\mathbb{R}^n)$ of functions of bounded deformation, i.e., of $\mathrm{L}^1(\mathbb{R}^n,\mathbb{R}^n)$–maps u such that the symmetric part $\mathcal{E}u$ of their distributional gradient is a bounded measure. The situation in this case is significantly more complicated, since, for example, we have $\mathrm{BV}(\mathbb{R}^n,\mathbb{R}^n)\subsetneq\mathrm{BD}(\mathbb{R}^n)$ by the so–called Ornstein's Non–inequality [4, 8, 10]; equivalently, there are maps $u\in\mathrm{BD}(\mathbb{R}^n)$ for which the full distributional gradient $\mathrm{D}u$ is not a Radon measure, so one cannot easily retrieve the approximate gradient of u from the absolutely continuous part of $\mathcal{E}u$ with respect to $\mathcal{L}^n$. It is however possible to recover u from $\mathcal{E}u$ via convolution with a $(1-n)$–homogeneous kernel (cp. Lemma 2.1). HAJŁASZ used this observation and a Marcinkiewicz–type characterisation of approximate differentiability to show approximate differentiability $\mathcal{L}^n$–a.e. of BD–functions ([7, Cor. 1]). This result was improved in [2, Thm. 7.4] to L^1–differentiability $\mathcal{L}^n$–a.e. by AMBROSIO, COSCIA, and DAL MASO, using the precise Korn–Poincaré Inequality of KOHN [9]. It was only recently when ALBERTI, BIANCHINI, and CRIPPA generalized the approach in [7], obtaining L^q–differentiability of BD–maps for $1\leqslant q<1^*$ (see [1, Thm. 3.4, Prop. 4.3]). It is, however, unclear whether the critical exponent $q=1^*$ can be reached using the Calderón–Zygmund–type approach in [1].

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In the present paper, we settle the question in [1, Rk. 4.5.(v)] of optimal differentiability of BD–maps in the positive (see Corollary 1.2). Although reminiscent of the elaborate estimates in [2, Sec. 7], our proof is rather straightforward. The key observation is to replace KOHN's Poincaré–Korn Inequality with the more abstract Korn–Sobolev Inequality due to STRANG and TEMAM [12, Prop. 2.4], combined with ideas developed recently by the authors in [6]. In fact, we shall prove $\mathrm{L}^{n/(n-1)}$–differentiability of maps of bounded $\mathbb{A}$–variation (as introduced in [3, Sec. 2.2]), provided that $\mathbb{A}$ has finite dimensional null–space.

To formally state our main result, we pause to introduce some terminology and notation. Let $\mathbb{A}$ be a linear, first order, homogeneous differential operator with constant coefficients on $\mathbb{R}^n$ from V to W, i.e.,

$$\mathbb{A}u = \sum_{j=1} A_j \partial_j u, \qquad u\colon \mathbb{R}^n \to V, \tag{1.1}$$

where $A_j \in \mathscr{L}(V, W)$ are fixed linear mappings between two finite dimensional real vector spaces V and W. For an open set $\Omega \subset \mathbb{R}^n$, we define $\mathrm{BV}^{\mathbb{A}}(\Omega)$ as the space of $u \in \mathrm{L}^1(\Omega, V)$ such that $\mathbb{A}u$ is a W–valued Radon measure. We say that $\mathbb{A}$ has *FDN* (finite dimensional null–space) if the vector space $\{u \in \mathscr{D}'(\mathbb{R}^n, V)\colon \mathbb{A}u = 0\}$ is finite dimensional. Using the main result in [6, Thm. 1.1], we will prove that FDN is sufficient to obtain a Korn–Sobolev–type inequality

$$\left(\fint_{\mathrm{B}_r} |u - \pi_{\mathrm{B}_r} u|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{n-1}{n}} \leqslant cr \fint_{\mathrm{B}_r} |\mathbb{A}u| \,\mathrm{d}x \tag{1.2}$$

for all $u \in \mathrm{C}^\infty(\bar{\mathrm{B}}_r, V)$. Here π denotes a suitable bounded projection on the null–space of $\mathbb{A}$, as described in [3, Sec. 3.1]. This is our main ingredient to prove the following:

Theorem 1.1. *Let $\mathbb{A}$ as in (1.1) have FDN, $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$. Then u is $\mathrm{L}^{n/(n-1)}$–differentiable at x for $\mathcal{L}^n$–a.e. $x \in \mathbb{R}^n$.*

Our example of interest is $\mathrm{BD} := \mathrm{BV}^{\mathcal{E}}$, where $\mathcal{E}u := \left(\mathrm{D}u + (\mathrm{D}u)^{\mathsf{T}}\right)/2$ for $u\colon \mathbb{R}^n \to \mathbb{R}^n$. It is well known that the null–space of $\mathcal{E}$ consists of rigid motions, i.e., affine maps of anti–symmetric gradient. In particular, $\mathcal{E}$ has FDN.

Corollary 1.2. *Let $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Then u is $\mathrm{L}^{n/(n-1)}$–differentiable $\mathcal{L}^n$–a.e.*

This paper is organized as follows: In Section 2 we collect some notation and definitions, mainly those of approximate and L^p–differentiability, present the main result in [1], collect a few results on $\mathbb{A}$–weakly differentiable functions from [3, 6], and prove the inequality (1.2). In Section 3 we give a brief proof of Theorem 1.1.

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2. Preliminaries

An operator $\mathbb{A}$ as in (1.1) can also be seen as $\mathbb{A}u = A(\mathrm{D}u)$ for $u\colon \mathbb{R}^n \to V$, where $A \in \mathscr{L}(V \otimes \mathbb{R}^n, W)$. We recall that such an operator has a Fourier symbol map

$$\mathbb{A}[\xi]v = \sum_{j=1}^n \xi_j A_j v,$$

defined for $\xi \in \mathbb{R}^n$ and $v \in V$. An operator $\mathbb{A}$ is said to be *elliptic* if and only if for all non–zero ξ, the maps $\mathbb{A}[\xi] \in \mathscr{L}(V,W)$ are injective. By considering the maps

$$u_f(x) := f(x\cdot\xi)v$$

for functions $f \in \mathrm{C}^1(\mathbb{R})$, it is easy to see that if $\mathbb{A}$ has FDN, then $\mathbb{A}$ is necessarily elliptic. Ellipticity is in fact equivalent with one–sided invertibility of $\mathbb{A}$ in Fourier space; more precisely, the equation $\mathbb{A}u = f$ can be uniquely solved for $u \in \mathscr{S}(\mathbb{R}^n, V)$ whenever $f \in \mathscr{S}(\mathbb{R}^n, W) \cap \operatorname{im}\mathbb{A}$. One has:

Lemma 2.1. *Let $\mathbb{A}$ be elliptic. There exists a convolution kernel $K^{\mathbb{A}} \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(W,V))$ which is $(1-n)$–homogeneous such that $u = K^{\mathbb{A}} * \mathbb{A}u$ for all $u \in \mathscr{S}(\mathbb{R}^n, V)$.*

For a proof of this fact, see, e.g., [6, Lem. 2.1]. We next define, for open $\Omega \subset \mathbb{R}^n$ (often a ball $\mathrm{B}_r(x)$), the space

$$\mathrm{BV}^{\mathbb{A}}(\Omega) := \{u \in \mathrm{L}^1(\Omega, V) \colon \mathbb{A}u \in \mathcal{M}(\Omega, W)\}$$

of maps of bounded $\mathbb{A}$–variation, which is a Banach space under the obvious norm. By the Radon–Nikodym Theorem $\mathbb{A}u$ has the decomposition

$$\mathbb{A}u = \mathbb{A}^{ac}u\mathcal{L}^n \llcorner \Omega + \mathbb{A}^s u := \frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathcal{L}^n}\mathcal{L}^n \llcorner \Omega + \frac{\mathrm{d}\mathbb{A}^s u}{\mathrm{d}|\mathbb{A}^s u|}|\mathbb{A}^s u|$$

with respect to $\mathcal{L}^n$. Here $|\cdot|$ denotes the total variation semi–norm. We next see that ellipticity of $\mathbb{A}$ implies sub–critical L^p–differentiability. We denote averaged integrals by $\fint_\Omega := \mathcal{L}^n(\Omega)^{-1}\int_\Omega$ or by $(\cdot)_{x,r}$ if $\Omega = \mathrm{B}_r(x)$, the ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$.

Definition 2.2. *A measurable map $u \colon \mathbb{R}^n \to V$ is said to be*

- *approximately differentiable at $x \in \mathbb{R}^n$ if there exists a matrix $M \in V \otimes \mathbb{R}^n$ such that*
$$\operatorname*{ap\,lim}_{y\to x} \frac{|u(y) - u(x) - M(y-x)|}{|y-x|} = 0;$$
- *L^p–differentiable at $x \in \mathbb{R}^n$, $1 \leqslant p < \infty$ if there exists a matrix $M \in V \otimes \mathbb{R}^n$ such that*
$$\left(\fint_{\mathrm{B}_r(x)} |u(y) - u(x) - M(y-x)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} = o(r)$$
as $r \downarrow 0$.

We say that $\nabla u(x) := M$ is the approximate gradient of u at x.

We should also recall that

$$v = \operatorname*{ap\,lim}_{y\to x} u(y) \iff \forall\varepsilon > 0,\ \lim_{r\downarrow 0} r^{-n}\mathcal{L}^n\left(\{y \in \mathrm{B}_r(x) \colon |u(y) - v| > \varepsilon\}\right) = 0,$$

where $x \in \mathbb{R}^n$ and $u \colon \mathbb{R}^n \to V$ is measurable. In the terminology of [1, Sec. 2.2], we can alternatively say that u is L^p–differentiable at x if

$$u(y) = \nabla u(x)(y-x) + u(x) + R_x(y), \tag{2.1}$$

where $(|R_x|^p)_{x,r} = o(r^p)$ as $r \downarrow 0$. We will refer to the decomposition (2.1) as a first order L^p–Taylor expansion of u about x.

Theorem 2.3 ([1, Thm. 3.4])**.** *Let $K \in \mathrm{C}^2(\mathbb{R}^n \setminus \{0\})$ be $(1-n)$–homogeneous, and $\mu \in \mathcal{M}(\mathbb{R}^n)$ be a bounded measure. Then $u := K * \mu$ is L^p–differentiable $\mathcal{L}^n$–a.e. for all $1 \leqslant p < n/(n-1)$.*

As a consequence of Lemma 2.1 and Theorem 2.3, we have that if $\mathbb{A}$ is elliptic, then maps in $\mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ are L^p–differentiable $\mathscr{L}^n$–a.e. for $1 \leqslant p < n/(n-1)$ (cp. Lemma 3.1). Ellipticity, however, is insufficient to reach the critical exponent. In Theorem 1.1, we show that FDN is a sufficient condition for the critical $\mathrm{L}^{n/(n-1)}$–differentiability. The following is essentially proved in [11], and is discussed at length in [3, 6]. We will, however, sketch an elementary proof for the interested reader.

Lemma 2.4. *Let $\mathbb{A}$ as in (1.1) have FDN. Then there exists $l \in \mathbb{N}$ such that null–space elements of $\mathbb{A}$ are polynomials of degree at most l.*

Sketch. One can show by standard arguments that if $\mathbb{A}$ is elliptic and $\mathbb{A}u = 0$ in $\mathscr{D}'(\mathbb{R}^n, V)$, then u is in fact analytic. If u is not a polynomial, then one can write u as an infinite sum of homogeneous polynomials and identify coefficients, thereby obtaining infinitely many linearly independent (homogeneous) polynomials in the null–space of $\mathbb{A}$. Then the kernel consists of polynomials, which must have a maximal degree, otherwise $\mathbb{A}$ fails to have FDN. $\square$

We next provide a Sobolev–Poincaré–type inequality which, in the $\mathbb{A}$–setting, follows from the recent work [6] and is the main ingredient in the proof of Theorem 1.1. Following [3, Sec. 3.1], we define for $\mathbb{A}$ with FDN, $\pi_B\colon \mathrm{C}^\infty \cap \mathrm{BV}^{\mathbb{A}}(\mathrm{B}) \to \ker\mathbb{A} \cap \mathrm{L}^2(\mathrm{B}, V)$ as the L^2–projection onto $\ker\mathbb{A}$.

Proposition 2.5 (Poincaré–Sobolev–type Inequality)**.** *Let $\mathbb{A}$ as in (1.1) have FDN. Then (1.2) holds. Moreover, there exists $c > 0$ such that*

$$\left(\fint_{\mathrm{B}_r(x)} |u - \pi_{\mathrm{B}_r(x)}u|^{\frac{n}{n-1}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} \leqslant cr^{1-n}|\mathbb{A}u|(\overline{\mathrm{B}_r(x)}).$$

for all $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $r > 0$.

Proof. By smooth approximation ([3, Thm. 2.8]), it suffices to prove (1.2). Since $\pi_{\mathrm{B}_r(x)}$ is linear, we can assume that $r = 1$, $x = 0$. The result then follows by scaling and translation. We abbreviate $\mathrm{B} := \mathrm{B}_1(0)$. By [6, Thm. 1.1] we have that

$$\left(\int_{\mathrm{B}} |u - \pi_{\mathrm{B}}u|^{\frac{n}{n-1}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} \leqslant c\left(\int_{\mathrm{B}} |\mathbb{A}u| + |u - \pi_{\mathrm{B}}u|\,\mathrm{d}y\right) \leqslant c\int_{\mathrm{B}} |\mathbb{A}u|\,\mathrm{d}y,$$

where for the second estimate we use the Poincaré–type inequality in [3, Thm. 3.2]. The proof is complete. $\square$

We conclude this section with a simple technical Lemma:

Lemma 2.6. *Let $l \in \mathbb{N}$. There exists a constant $c > 0$ independent of any ball $\mathrm{B} \subset \mathbb{R}^n$ such that*

$$\sup_{y\in\mathrm{B}} |P(y)| \leqslant c\fint_{\mathrm{B}} |P(y)|\,\mathrm{d}y \tag{2.2}$$

for any polynomial of degree at most l.

Proof. The space the polynomials of degree at most l restricted on the unit ball is finite dimensional, hence the L^∞ and L^1 norms are equivalent. In particular, (2.2) holds for $\mathrm{B} = \mathrm{B}_1(0)$. Consider $\mathrm{B} := \mathrm{B}_r(x)$. Then

$$\sup_{y\in\mathrm{B}} |P(y)| = \sup_{z\in\mathrm{B}_1(0)} |P(x+rz)| \leqslant c\fint_{\mathrm{B}_1(0)} |P(x+rz)|\,\mathrm{d}z = c\fint_{\mathrm{B}_r(x)} |P(y)|\,\mathrm{d}y,$$

since $P(x + r\,\cdot)$ are polynomials of degree at most l. The proof is complete. $\square$

3. Proof of Theorem 1.1

We begin by proving sub–critical L^p–differentiability of $u \in \mathrm{BV}^{\mathbb{A}}$ for elliptic $\mathbb{A}$ (cp. [7, Thm. 5]). We also provide a formula that enables us to retrieve the absolutely continuous part of $\mathbb{A}u$ from the approximate gradient. This formula respects the algebraic structure of $\mathbb{A}$, generalizing the result for BD in [2, Rk. 7.5].

Lemma 3.1. *If $\mathbb{A}$ is elliptic, then any map $u \in \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ is L^p–differentiable $\mathcal{L}^n$–a.e. for all $1 \leqslant p < n/(n-1)$. Moreover, we have that*

$$\frac{\mathrm{d}\mathbb{A}u}{\mathrm{d}\mathcal{L}^n}(x) = A(\nabla u(x)) \tag{3.1}$$

for $\mathcal{L}^n$–a.e $x \in \mathbb{R}^n$.

Proof. By Lemma 2.1, we can write the components $u_i = K^{\mathbb{A}}_{ij} * (\mathbb{A}u)_j$, where summation over repeated indices is adopted. We then note that $K^{\mathbb{A}}_{ij}$ satisfies the assumptions of Theorem 2.3, hence each component u_i is L^p–differentiable $\mathcal{L}^n$–a.e. for $1 \leqslant p < n/(n-1)$.

We next let $u \in \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ be a Lebesgue point of u and $\mathbb{A}^{ac}u$, and also a point of L^1–differentiability of u. We also consider a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of standard mollifiers, i.e., $\eta_1 \in \mathrm{C}^\infty_c(\mathrm{B}_1(0))$ is radially symmetric and has integral equal to 1 and $\eta_\varepsilon(y) = \varepsilon^{-n}\eta_1(x/\varepsilon)$. Finally, we write $u_\varepsilon := u * \eta_\varepsilon$ and employ the Taylor expansion (2.1) to compute

$$\begin{aligned}
\nabla u_\varepsilon(x) &= \int_{\mathrm{B}_\varepsilon(x)} u(y) \otimes \nabla_x \eta_\varepsilon(x-y)\,\mathrm{d}y \\
&= -\int_{\mathrm{B}_\varepsilon(x)} (\nabla u(x)(y-x) + u(x) + R_x(y)) \otimes \nabla_y \eta_\varepsilon(y-x)\,\mathrm{d}y \\
&= \int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(y-x)\nabla u(x)\,\mathrm{d}y - \int_{\mathrm{B}_\varepsilon(x)} R_x(y) \otimes \nabla_y \eta_\varepsilon(y-x)\,\mathrm{d}y \\
&= \nabla u(x) + \int_{\mathrm{B}_\varepsilon(x)} R_x(y) \otimes \nabla_x \eta_\varepsilon(x-y)\,\mathrm{d}y,
\end{aligned}$$

where we used integration by parts to establish the third equality. Since

$$\|\nabla_x \eta(x-\cdot)\|_\infty = \varepsilon^{-(n+1)}\|\nabla\eta_1\|_\infty,$$

we have that $|\nabla u_\varepsilon(x) - \nabla u(x)| \leqslant c(n,\eta_1)\varepsilon^{-1}(|R_x|)_{x,\varepsilon} = o(1)$ as x is a point of L^1–differentiability of u. In particular, $\nabla u_\varepsilon \to \nabla u$ $\mathcal{L}^n$–a.e., so that $\mathbb{A}u_\varepsilon \to A(\nabla u)$ $\mathcal{L}^n$–a.e. To establish (3.1), we will show that $\mathbb{A}u_\varepsilon \to \mathbb{A}^{ac}u$ $\mathcal{L}^n$–a.e. Using only that u is a distribution, one easily shows that $\mathbb{A}u_\varepsilon = \mathbb{A}u * \eta_\varepsilon$, so that

$$\begin{aligned}
\mathbb{A}u_\varepsilon(x) - \mathbb{A}^{ac}u(x) &= \mathbb{A}^{ac}u * \eta_\varepsilon(x) - \mathbb{A}^{ac}u(x) + \mathbb{A}^s u * \eta_\varepsilon(x) \\
&= \int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)\,(\mathbb{A}^{ac}u(y) - \mathbb{A}^{ac}u(x))\,\mathrm{d}y \\
&\quad + \int_{\mathrm{B}_\varepsilon(x)} \eta_\varepsilon(x-y)\,\mathrm{d}\mathbb{A}^s u(y).
\end{aligned}$$

Using the fact that $\|\eta_\varepsilon(x-\cdot)\|_\infty = \varepsilon^{-n}\|\eta_1\|_\infty$ and Lebesgue differentiation, the proof is complete. □

Remark 3.2 (Insufficiency of ellipticity)**.** *Consider v as in* [1, Prop. 4.2] *with $n = 2$. One shows by direct computation that $v \in \mathrm{BV}^{\partial}(\mathbb{R}^2)$, where the Wirtinger* *derivative*

$$\partial u := \frac{1}{2}\begin{pmatrix} \partial_1 u_1 + \partial_2 u_2 \\ \partial_2 u_1 - \partial_1 u_2 \end{pmatrix}$$

is easily seen to be elliptic (computation). However, it is shown in [1, Rk. 4.5.(iv)] *that there are maps* $v \in \mathrm{BV}^{\partial}(\mathbb{R}^2)$ *which are not* L^2*–differentiable.*

In turn, the stronger FDN condition is sufficient for L^{1^*}–differentiability:

Proof of Theorem 1.1. Let $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ that is a Lebesgue point of $\mathbb{A}u$ such that

$$\fint_{\mathrm{B}_r(x)} |u(y) - u(x) - \nabla u(x)(y-x)|\,\mathrm{d}y = o(r) \tag{3.2}$$

as $r \downarrow 0$. By Lemma 3.1 for $p = 1$, such points exist $\mathcal{L}^n$–a.e. Here $\nabla u(x)$ denotes the approximate gradient of u at x. We define $v(y) := u(y) - u(x) - \nabla u(x)(y-x)$ for $y \in \mathbb{R}^n$. We aim to show that

$$\left(\fint_{\mathrm{B}_r(x)} |v(y)|^{\frac{n}{n-1}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} = o(r) \tag{3.3}$$

as $r \downarrow 0$. Firstly, we remark that the integral in (3.3) is well–defined for $r > 0$, as v is the sum of an affine and a $\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$–map; the latter is $\mathrm{L}^{n/(n-1)}_{\mathrm{loc}}$–integrable, e.g., by [6, Thm. 1.1]. Next, we abbreviate $\pi_r v := \pi_{\mathrm{B}_r(x)} v$ and use Proposition 2.5 to estimate:

$$\begin{aligned} \left(\fint_{\mathrm{B}_r(x)} |v|^{1^*}\,\mathrm{d}y\right)^{\frac{1}{1^*}} &\leqslant \left(\fint_{\mathrm{B}_r(x)} |v - \pi_r v|^{1^*}\,\mathrm{d}y\right)^{\frac{1}{1^*}} + \left(\fint_{\mathrm{B}_r(x)} |\pi_r v|^{1^*}\,\mathrm{d}y\right)^{\frac{1}{1^*}} \\ &\leqslant cr\frac{|\mathbb{A}v|(\overline{\mathrm{B}_r(x)})}{r^n} + \left(\fint_{\mathrm{B}_r(x)} |\pi_r v|^{\frac{n}{n-1}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} =: \mathbf{I}_r + \mathbf{II}_r. \end{aligned}$$

To deal with $\mathbf{I}_r$, first note that $\mathbb{A}v = \mathbb{A}u - A(\nabla u(x))$ (the latter term is obtained by classical differentiation of an affine map). By (3.1), we obtain $\mathbb{A}v = \mathbb{A}u - \mathbb{A}^{ac}u(x)$, so $\mathbf{I}_r = o(r)$ as $r \downarrow 0$ by Lebesgue differentiation for Radon measures. To bound $\mathbf{II}_r$, by Lemma 2.4, we can use (2.2) to get that

$$\left(\fint_{\mathrm{B}_r(x)} |P|^{\frac{n}{n-1}}\,\mathrm{d}y\right)^{\frac{n-1}{n}} \leqslant c\fint_{\mathrm{B}_r(x)} |P|\,\mathrm{d}y,$$

so that we have $\mathbf{II}_r \leqslant c(|\pi_r v|)_{x,r}$. We claim that

$$\fint_{\mathrm{B}_r(x)} |\pi_r v|\,\mathrm{d}y \leqslant c\fint_{\mathrm{B}_r(x)} |v|\,\mathrm{d}y, \tag{3.4}$$

which suffices to conclude by (3.2), and (3.3). Though elementary and essentially present in [3, Sec. 3.1], the proof of (3.4) is delicate and we present a careful argument. We write

$$\pi_r v = \sum_{j=1}^{d} \langle v, e_j^r\rangle e_j^r,$$

where the inner product is taken in L^2 and $\{e_j^r\}_{j=1}^d$ is a (finite) orthonormal basis of $\ker\mathbb{A}\cap\mathrm{L}^2(\mathrm{B}_r(x),V)$. By (2.2) and Cauchy–Schwarz inequality we have

$$\sup_{y\in\mathrm{B}_r(x)}|e_j^r(y)|\leqslant c\left(\fint_{\mathrm{B}_r(x)}|e_j^r|^2\,\mathrm{d}y\right)^{\frac{1}{2}}=cr^{-\frac{n}{2}},$$

so that

$$\fint_{\mathrm{B}_r(x)}|\pi_r v|\,\mathrm{d}y\leqslant\sum_{j=1}^d\fint_{\mathrm{B}_r(x)}\int_{\mathrm{B}_r(x)}|v|\,\mathrm{d}z\,\mathrm{d}y\|e_j^r\|^2_{\mathrm{L}^\infty(\mathrm{B}_r(x),V)}\leqslant cr^{-n}\int_{\mathrm{B}_r(x)}|v|\,\mathrm{d}z,$$

which yields (3.4) and concludes the proof. □

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