CONTINUITY POINTS VIA RIESZ POTENTIALS FOR $\mathbb{C}$-ELLIPTIC OPERATORS

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ABSTRACT. We establish a Riesz potential criterion for Lebesgue continuity points of functions of bounded $\mathbb{A}$-variation, where $\mathbb{A}$ is a $\mathbb{C}$-elliptic differential operator of arbitrary order. This result might even be of interest for classical functions of bounded variation.

1. INTRODUCTION

Functions of bounded variation are a vastly studied subject, mainly as they form the natural function space framework for a variety of variational problems. Hence it is particularly important to understand their fine properties, and a wealth of contributions on this theme is available, cf. AMBROSIO et al. [2] and the references therein. When dealing with *full* gradients, powerful tools such as the coarea formula are available, facilitating the proofs of key results in this context such as the absolute continuity of Du for $\mathscr{H}^{n-1}$ or in the study of Lebesgue discontinuity points.

Various variational problems, however, require to work with more general differential operators than the usual gradient, see [9] for a comprehensive account of problems from elasticity or plasticity. To provide a unifying approach to this topic, let $\mathbb{A}$ be a k-th order, homogeneous, constant-coefficient differential operator on $\mathbb{R}^n$ between the two finite dimensional real vector spaces V and W. By this we understand that $\mathbb{A}$ has a representation

$$\mathbb{A}u = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=k}} \mathbb{A}_\alpha \partial^\alpha u, \tag{1.1}$$

where $\mathbb{A}_\alpha \in \mathscr{L}(V;W)$ are fixed linear maps; note that ∂^α acts compontentwisely on a function $u : \mathbb{R}^n \to V$. If $\mathbb{A}$ is *elliptic* (cf. HÖRMANDER [13] or SPENCER [22]), meaning that the Fourier symbol

$$\mathbb{A}[\xi] = \sum_{|\alpha|=k} \xi_\alpha \mathbb{A}_\alpha \colon V \to W \tag{1.2}$$

is injective for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then elementary Fourier multiplier techniques establish that for each $1 < p < \infty$ there exists $c_p > 0$ such that there holds

$$\|D^k u\|_{L^p(\mathbb{R}^n)} \le c_p \|\mathbb{A}u\|_{L^p(\mathbb{R}^n)} \qquad \text{for all } u \in C_c^\infty(\mathbb{R}^n; V). \tag{1.3}$$

Inequalities of this type are usually referred to as KORN-type inequalities, cf. [8, 17, 4] for instance. By a foundational result of ORNSTEIN [18], estimate (1.3) does not persist for $p = 1$ in general. Indeed, by the sharp version as recently established by KIRCHHEIM & KRISTENSEN [15, 16], validity of (1.3) for $p = 1$ is equivalent to the existence of some $T \in \mathscr{L}(W; V \odot^k \mathbb{R}^n)$ such that $D^k = T \circ \mathbb{A}$. In this case,

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however, (1.3) trivalises, leading to the informal metaprinciple that *there are no non-trivial L^1-estimates.*

Let $u \in L^1(\mathbb{R}^n; V)$. Given a differential operator of the form (1.1), we say that u is of *bounded* $\mathbb{A}$*-variation* and write $u \in \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ if and only if the distributional differential expression $\mathbb{A}u$ can be represented by a finite W-valued Radon measure, denoted $\mathbb{A}u \in \mathscr{M}(\mathbb{R}^n; W)$. The space $\mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}$ then is defined in the obvious way. This class of function spaces has been introduced in [5, 10], and by ORNSTEIN's Non-Inequality, we have $\mathrm{BV} \subsetneq \mathrm{BV}^{\mathbb{A}}$ in general. As a major obstruction in the study of $\mathrm{BV}^{\mathbb{A}}$-maps, the failure of (1.3) for $p = 1$ equally rules out the use of full gradient techniques. Still, as is by now well-known, several properties of BV-maps can be proven to hold for $\mathrm{BV}^{\mathbb{A}}$-maps, too, and we refer the reader to [24, 3, 25, 5, 10, 11, 12, 19, 20] for a comprehensive list of results in this area.

Based on SMITH [21], in [5] BREIT and the authors isolated a key property of first order differential operators $\mathbb{A}$ to yield *boundary trace embeddings* $\mathrm{BV}^{\mathbb{A}}(\Omega) \hookrightarrow L^1(\partial\Omega; V)$, namely $\mathbb{C}$*-ellipticity*. We say that a differential operator of the form (1.1) is $\mathbb{C}$-elliptic provided the complexified Fourier symbol

$$\mathbb{A}[\xi]\colon V + \mathrm{i}\,V \to W + \mathrm{i}\,W \qquad \text{is injective for all } \xi \in \mathbb{C}^n \setminus \{0\}.$$

As a particular consequence of [5, Thm. 1.1], $\mathbb{C}$-elliptic differential operators have finite dimensional nullspace (in $\mathscr{D}'(\mathbb{R}^n; V)$) consisting of polynomials of a fixed maximal degree exclusively. As is well-known from the classical BV-theory, interior traces are instrumental for a description of the jump parts and hence the set of Lebesgue discontinuity points. This is the starting point for the present paper, where we aim to introduce Riesz potential techniques in the study of $\mathrm{BV}^{\mathbb{A}}$-functions for the particular case of $\mathbb{C}$-elliptic differential operators.

It is easy to see that finiteness of the fractional maximal operator $\mathcal{M}_k(\mathbb{A}u)$ with

$$\mathcal{M}_k\mu(x) := \sup_{B \ni x} \frac{|\mu|(B(x,r))}{r^{n-k}}, \qquad \mu \in \mathcal{M}(\mathbb{R}^n; W),\ x \in \mathbb{R}^n,$$

cannot yield a criterion for Lebesgue continuity points. In fact, consider $u = \mathbb{1}_{\{x : |x| < 1, x_n > 0\}}$ and $k = 1$, for which $\mathcal{M}_1(Du)(x) < \infty$ for any $x \in \partial B(0,1)$. Opposed to this, the main result of the present paper is that the Riesz potentials are sufficiently powerful to *detect Lebesgue continuity points* indeed. Given a Radon measure $\mu \in \mathscr{M}(\mathbb{R}^n; W)$ and $s > 0$, we define the *Riesz potential of* μ *of order* s by

$$\mathcal{I}_s(\mu)(x_0) := \int_{B(x_0,r)} \frac{\mathrm{d}|\mu|(y)}{|x_0 - y|^{n-s}} < \infty, \qquad x_0 \in \mathbb{R}^n. \tag{1.4}$$

Let S_u denote the set of *non-Lebesgue points* of a map $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n; V)$.

Theorem 1.1. *Let $n \geq 2$, $k \geq 1$ and let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form* (1.1)*. Then the following hold:*

(a) If $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$ are such that for some $r > 0$ there holds

$$\mathcal{I}_k(\mathbb{A}u \llcorner B(x_0,r))(x_0) < \infty, \tag{1.5}$$

then x_0 is a Lebesgue point for u, i.e. $x_0 \in S_u^{\complement}$.

(b) If $k = n$, then any $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$ has a continuous representative. If $x \in \mathbb{R}^n$ and $r > 0$, then for every $y \in B(x, r/2)$ the continuous representative satisfies

$$|u(x) - u(y)| \leq c\,|\mathbb{A}u|(B \setminus \{x\}) + c\,\frac{|x-y|}{r} \fint_B |u - \langle u \rangle_B|\,\mathrm{d}z, \tag{1.6}$$

where $B := B(x,r)$.

(c) If $k>n$, then any $u\in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$ has a C^{n-k} representative. If $x\in\mathbb{R}^n$ and $r>0$, then for every $y\in B(x,r/2)$ the continuous representative satisfies

$$\begin{aligned}|(\nabla^{k-n}u)(x)-(\nabla^{k-n}u)(y)| &\leq c\,|\mathbb{A}u|(B\setminus\{x\})\\ &\quad + c\,\frac{|x-y|}{r}\fint_B |\nabla^{k-n}u-\langle\nabla^{k-n}u\rangle_B|\,\mathrm{d}z,\end{aligned} \tag{1.7}$$

where $B:=B(x,r)$.

Throughout the paper we consider the case $n\geq 2$. Indeed, for $n=1$ the only elliptic operators are of the form $a\frac{\mathrm{d}}{\mathrm{d}x}$ for $a\in\mathbb{R}\setminus\{0\}$. Hence, if $n=1$, $\mathrm{BV}^{\mathbb{A}}(\mathbb{R})=\mathrm{BV}(\mathbb{R})$ for all elliptic operators $\mathbb{A}$ and so (b) of Theorem 1.1 fails. The case $n\geq 2$ differs from the case $n=1$ in the fact that annuli are connected for $n\geq 2$ and allow therefore to deduce a Poincaré-type inequality on annuli, cf. Corollary 2.3. So from now on let $n\geq 2$.

Let us define, for $u\in\mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$,

$$\Sigma_u := \{x_0\in\mathbb{R}^n\colon\ \mathcal{I}_k(\mathbb{A}u\llcorner B(x_0,r))=\infty \text{ for all } r>0\}\,. \tag{1.8}$$

By Theorem 1.1 (a), $\Sigma_u^{\complement}\subset S_u^{\complement}$ and so $S_u\subset\Sigma_u$. From general measure and potential theoretic principles it then follows that the Hausdorff dimension of Σ_u and hence S_u cannot exceed $(n-k)$ for a $\mathbb{C}$-elliptic operator $\mathbb{A}$ and $u\in\mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$. If $k\geq n\geq 2$, then by Theorem 1.1(b) and Remark 3.3 more can be said: In this case, we have $\mathrm{BV}^{\mathbb{A}}(\Omega)\hookrightarrow C^{k-n}(\overline{\Omega};V)$ for every bounded Lipschitz domain. Thus, we obtain an independent proof *for the case of $\mathbb{C}$-elliptic operators* of a more general borderline embedding theory developed by Raita & Skorobogatova [20, Thm. 1.1, 1.3] for elliptic and cancelling operators. The case $k>n$ establishes how [19, Thm. 1.4] can be strengthened in the case of $\mathbb{C}$-elliptic operators and seems to be new.

Let us finally explain the structure of the paper. In Section 2 we gather notation and background facts on $\mathrm{BV}^{\mathbb{A}}$-functions and Poincaré type inequalities. In Section 3, we provide oscillation estimates for functions u in terms of the Riesz potentials of $\mathbb{A}u$, which will eventually yield the proof of Theorem 1.1 in Section 3.2.

2. Preliminaries

We start by fixing notation. Throughout, $B=B(x_0,r)$ denotes the open ball in $\mathbb{R}^n$ of radius $r>0$ centered at x_0. We also often use annuli of the form $\mathfrak{A}=B\setminus\lambda\overline{B}$ for some $\lambda\in[0,\frac12]$. For $s>0$ we denote by sB, resp. $s\mathfrak{A}$, the balls, resp. annuli, that are scaled by the factor s by keeping the center in place. As usual, we denote $\mathscr{L}^n$ or $\mathscr{H}^{n-1}$ the n-dimensional Lebesgue or $(n-1)$-dimensional Hausdorff measures, respectively, and sometimes abbreviate $|U|:=\mathscr{L}^n(U)$. Whenever B is an open ball, we use the equivalent notations

$$\langle u\rangle_B := \fint_B u\,\mathrm{d}x := \frac{1}{|B|}\int_B u\,\mathrm{d}x.$$

Given a finite dimensional real vector space X, the finite, X-valued Radon measures on the open set Ω are denoted $\mathscr{M}(\Omega;X)$. Also, given $\mu\in\mathscr{M}(\Omega;X)$ and Borel subsets A,U of Ω, we define $(\mu\llcorner A)(U):=\mu(A\cap U)$. Finally, by $c>0$ we denote a generic constant that might change from line to line and shall only be specified if its precise dependence on other parameters is required.

For the following, let $\mathbb{A}$ be a $\mathbb{C}$-elliptic differential operator of the form (1.1). We then record the following facts, retrievable from [5, 10]. $\mathbb{C}$-ellipticity of $\mathbb{A}$ implies that for any connected, open subset $\Omega\subset\mathbb{R}^n$ the nullspace $N(\mathbb{A};\Omega):=\{u\in\mathscr{D}'(\Omega;V)\colon\ \mathbb{A}u=0\}$ of $\mathbb{A}$ is finite dimensional and is a subset of the set of polynomials $\mathscr{P}_m(\Omega;V)$ of a fixed maximal degree $m\in\mathbb{N}$. As such, for any open, bounded and connected $\Omega \subset \mathbb{R}^n$, $N(\mathbb{A};\Omega) \subset L^2(\Omega;V)$ and we may define Π_Ω to be the L^2-orthogonal projection onto $N(\mathbb{A};\Omega)$.

Following Smith [21] or Kałamajska [14] we can represent every $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, where Ω is a star-shaped domain with respect to a ball B_Ω by a projection and a convolution of $\mathbb{A}u$ with a Riesz potential kernel. In particular, there exists $m \in \mathbb{N}_0$ depending on $\mathbb{A}$ and an integral kernel $\mathfrak{K}_\Omega \colon \Omega \times \Omega \to \mathscr{L}(W;V)$ which is C^∞ off the diagonal $\{x=y\}$ and satisfies $|\partial_x^\alpha \partial_y^\alpha \mathfrak{K}_\Omega(x,y)| \leq c_{\alpha,\beta}\,|x-y|^{k-n-|\alpha|-|\beta|}$ such that

$$u(x) = \mathbb{P}^m_{B_\Omega} u(x) + \int\limits_{\operatorname{conv\,hull}(\overline{B_\Omega}\cup\{x\})} \mathfrak{K}_\Omega(x-y)\mathbb{A}u(y)\,\mathrm{d}y \tag{2.1}$$

for $\mathscr{L}^n$-a.e. $x \in \Omega$, where $\mathbb{P}^m_{B_\Omega} u$ denote the averaged Taylor polynomial of order m with respect to the ball B_Ω and $\operatorname{conv\,hull}(\overline{B_\Omega}\cup\{x\})$ is the closed convex hull of $B_\Omega \cup \{x\}$.

It follows from this representation and the property that $\nabla^\ell \mathbb{P}^m_{B_\Omega} = \mathbb{P}^{m-\ell}_{B_\Omega}\nabla^\ell$ that for every $l = 0,\dots,k-1$ there exists $\mathfrak{K}^\ell_\Omega \colon \Omega\times\Omega \to \mathscr{L}(W;V)$ which is C^∞ off the diagonal $\{x=y\}$ and satisfies $|\partial_x^\alpha \partial_y^\alpha \mathfrak{K}^\ell_\Omega(x,y)| \leq c_{\alpha,\beta}\,|x-y|^{k-\ell-n-|\alpha|-|\beta|}$ such that

$$\nabla^\ell u(x) = \mathbb{P}^{n-\ell}_{B_\Omega}\nabla^\ell u(x) + \int\limits_{\operatorname{conv\,hull}(\overline{B_\Omega}\cup\{x\})} \mathfrak{K}^\ell_\Omega(x-y)\mathbb{A}u(y)\,\mathrm{d}y. \tag{2.2}$$

Working from this representation, we infer the usual Poincaré-type estimates:

Lemma 2.1 (Poincaré for star-shaped domains)**.** *Let Ω be star-shaped domain with respect to the ball B_Ω with radius r_B and* $\operatorname{diam}(\Omega) \leq c\,r_B$*. Let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form* (1.1)*. Then there exists a constant $c = c(\mathbb{A}) > 0$ such that*

$$\sum_{\ell=0}^{k-1} \fint_\Omega r_B^\ell |\nabla^\ell u - \nabla^\ell \mathbb{P}^m_{B_\Omega} u|\,\mathrm{d}x \leq c\,r^k \frac{|\mathbb{A}u|(\Omega)}{|\Omega|}. \tag{2.3}$$

holds for all $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$. Recall that $\nabla^\ell \mathbb{P}^m_{B_\Omega} = \mathbb{P}^{m-\ell}_{B_\Omega}\nabla^\ell$.

In the following we show how to replace the averaged Taylor polynomial in this formula by the projection Π_B.

Corollary 2.2. *Under the assumptions of Lemma 2.1 there holds*

$$\sum_{\ell=0}^{k-1} \fint_\Omega r_B^\ell |\nabla^\ell u - \nabla^\ell \Pi_B u|\,\mathrm{d}x \leq c\,r^k \frac{|\mathbb{A}u|(\Omega)}{|\Omega|}. \tag{2.4}$$

Proof. It remains to show

$$\mathrm{I} := \sum_{\ell=0}^{k-1} \fint_\Omega r_\Omega^\ell |\nabla^\ell(\mathbb{P}^m_{B_\Omega}u - \Pi_B u)|\,dx \leq c\,r^k \frac{|\mathbb{A}u|(\Omega)}{|\Omega|}.$$

Since $N(\mathbb{A}) \subset \mathscr{P}_m$ we have $\mathbb{P}^m_{B_\Omega}u - \Pi_B u = \mathbb{P}^m_{B_\Omega}u - \Pi_B \mathbb{P}^m_{B_\Omega}u$. Moreover, for any $p \in \mathscr{P}_m$ there holds

$$\sum_{\ell=0}^{k-1} \fint_\Omega r_\Omega^\ell |\nabla^\ell p - \nabla^\ell p|\,dx \leq c \fint_\Omega |\mathbb{A}p|\,dx,$$

since both sides define a norm on the finite dimensional space $\mathscr{P}_l/N(\mathbb{A})$ and vanish on $N(\mathbb{A})$. Thus,

$$\mathrm{I} \leq \sum_{\ell=0}^{k-1} \fint_\Omega r_\Omega^\ell |\nabla^\ell(\mathbb{P}^m_{B_\Omega}u - \Pi_B \mathbb{P}^m_{B_\Omega}u)|\,dx$$

$$
\begin{aligned}
&\leq c \fint_B |\mathbb{A}(\mathbb{P}^m_{B_\Omega} u - \Pi_B \mathbb{P}^m_{B_\Omega} u)|\,dx\\
&= c \fint_\Omega |\mathbb{P}^{m-k}_{B_\Omega} \mathbb{A} u|\,dx.\\
&\leq c \fint_\Omega |\mathbb{A} u|\,dx
\end{aligned}
$$

using that $\nabla^\ell \mathbb{P}^m_{B_\Omega} = \mathbb{P}^{m-\ell}_{B_\Omega} \nabla^\ell$ and the L^1-stability of $\mathbb{P}^{m-k}_{B_\Omega}$. The proof of the corollary is complete. □

We also need the Poincaré-type inequality for annuli and punctured balls.

Corollary 2.3 (Poincaré annuli). *Let $n \geq 2$. Let B be ball with radius r_B. Let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form (1.1). Then there exists a constant $c = c(\mathbb{A}) > 0$ such that the following holds. Let Ω be the annulus $\mathfrak{A}_\lambda := B \setminus \lambda \overline{B}$ with $\lambda \in [0, \frac{1}{2}]$. Then*

$$
\sum_{\ell=0}^{k-1} \fint_\Omega r_B^\ell |\nabla^\ell u - \nabla^\ell \Pi_\Omega u|\,\mathrm{d}x \leq c\, r^k \frac{|\mathbb{A} u|(\Omega)}{|\Omega|}. \tag{2.5}
$$

holds for all $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$.

Proof. The estimate as in Lemma 2.1 involving $\nabla^\ell \mathbb{P}^m_{B_\Omega} u$ (where B_Ω is a suitable sub-ball of Ω) follows for Ω in fact by a standard argument and works in fact for any bounded Lipschitz domain. Since Ω can be written as the finite union of overlapping subdomains $\Omega_1, \dots, \Omega_N$ (with N depending only on n) which are star shaped with respect to a ball. These subdomains can be constructed, such that $\Omega_j \cap \Omega_{j+1}$ contain a ball B_j of size equivalent to Ω and Ω_j and Ω_{j+1} both are star shaped with respect to this ball B_j. The difference of the averaged Taylor polynomials on two consecutive balls B_j and B_{j+1} can be estimated again by Lemma 2.1. Now, we can change $\mathbb{P}^m_{B_\Omega}$ to Π_Ω exactly as in Corollary 2.2. □

Remark 2.4. *Based on the technique introduced in [7], the Poincaré-type inequalities of the above form can moreover be established for bounded John domains, a fact that we shall pursue elsewhere.*

Note that, Corollary 2.3 fails for $n = 1$. The problem is that the annuli are not connected for $n = 1$.

It is well known that there exists a constant $c = c(n, m, \dim(V)) > 0$ such that for all $q \in \mathscr{P}_m(\mathbb{R}^n; V)$ and all balls B there holds

$$
\frac{1}{c} \fint_B |q|\,\mathrm{d}x \leq \|q\|_{L^\infty(B)} \leq c \fint_B |q|\,\mathrm{d}x. \tag{2.6}
$$

Such estimate are usually called *inverse estimates.* In fact the estimate follows by the equivalence of all norms on finite dimensional spaces and scaling.

These inverse estimate on $N(\mathbb{A}) \subset \mathscr{P}_m$ (for bounded Ω) and the self-adjointness of Π_Ω allows in a standard way to extend Π_Ω to a $L^1(\Omega)$ with

$$
\fint_\Omega |\Pi_\Omega u|\,\mathrm{d}x \leq c \fint_\Omega |u|\,\mathrm{d}x. \tag{2.7}
$$

We later refer to this as the L^1-stability of Π_Ω.

We need the following inverse estimates for polynomials that vanish at the center of the ball.

Lemma 2.5. *Let $m \in \mathbb{N}_0$. Then there exists $c = c(m,n)$ such that for all balls B with center x_0, all $\lambda \in (0,1]$ and all $q \in \mathscr{P}_m(\mathbb{R}^n;V)$ with $q(x_0)=0$, we have*

$$\fint_{\lambda B} |q(x)|\,\mathrm{d}x \leq c\,\lambda \fint_B |q(x)|\,\mathrm{d}x.$$

Proof. By translation and scaling it suffices to establish the claim in the case $B = B(0,1)$. Because of $q(0)=0$ we may write $q(x) = x\cdot q_1(x)$. Thus

$$\fint_{\lambda B(0,1)} |q(x)|\,\mathrm{d}x = \fint_{\lambda B(0,1)} |x\cdot q_1(x)|\,\mathrm{d}x \leq \lambda \max_{B(0,1)} |q_1(x)|.$$

Now,

$$\max_{x\in B(0,1)} |q_1(x)| \eqsim \fint_{B(0,1)} |x||q_1(x)|\,\mathrm{d}x,$$

since both terms are norms on the finite dimensional space $\mathscr{P}_m(\mathbb{R}^n;V)$. Hence,

$$\fint_{\lambda B(0,1)} |q(x)|\,\mathrm{d}x \leq c\,\lambda \fint_{B(0,1)} |x||q_1(x)|\,\mathrm{d}x = c\,\lambda \fint_B |q|\,\mathrm{d}x,$$

where we have used in the last step inverse estimates for polynomials. □

3. Oscillation estimates and the proof of Theorem 1.1

3.1. **Oscillation estimates.** Throughout the entire section, let $n \geq 2$. We moreover tacitly assume $\mathbb{A}$ to be a k-th order $\mathbb{C}$-elliptic operator of the form (1.1) and let $u \in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, we fix $x_0 \in \mathbb{R}^n$, $r>0$ and put $B := B(x_0,r)$. Toward Theorem 1.1, we begin with the following lemma which allows to control oscillations of $u - \langle u\rangle_B$ be means of $u - \Pi_B u$.

Proposition 3.1. *There exists $c = c(\mathbb{A}) > 0$ such that for all $u \in L^1(B;V)$ there holds for each ball B and annulus $\mathfrak{A} = B \setminus \frac{1}{4}\overline{B}$*

$$\begin{aligned}\fint_{2^{-j}B} |u - \langle u\rangle_{2^{-j}B}|\,\mathrm{d}x &\leq 2^{-j}c \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}|\,\mathrm{d}x \\ &\quad + c\fint_{2^{-j}B} |u - \Pi_{2^{-j}B}u|\,\mathrm{d}x \\ &\quad + c\sum_{m=0}^{j} 2^{m-j} \fint_{2^{-m}\mathfrak{A}} |u - \Pi_{2^{-m}\mathfrak{A}}u|\,\mathrm{d}x.\end{aligned} \tag{3.1}$$

Moreover,

$$\begin{aligned}\fint_{2^{-j}\mathfrak{A}} |u - \langle u\rangle_{2^{-j}\mathfrak{A}}|\,\mathrm{d}x &\leq 2^{-j}c \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}|\,\mathrm{d}x \\ &\quad + c\sum_{m=0}^{j} 2^{m-j} \fint_{2^{-m}\mathfrak{A}} |u - \Pi_{2^{-m}\mathfrak{A}}u|\,\mathrm{d}x.\end{aligned} \tag{3.2}$$

Proof. We begin with the proof of (3.1). By routine means, we then find

$$\begin{aligned}\fint_{2^{-j}B} |u - \langle u\rangle_{2^{-j}B}|\,\mathrm{d}x &\leq 2 \fint_{2^{-j}B} |u(x) - (\Pi_{2^{-j}B}u)(x_0)|\,\mathrm{d}x \\ &\leq 2\fint_{2^{-j}B} |u - \Pi_{2^{-j}B}u|\,\mathrm{d}x \\ &\quad + 2\,\|\Pi_{2^{-j}B}u - \Pi_{2^{-j}\mathfrak{A}}u\|_{L^\infty(2^{-j}B)} \\ &\quad + 2\fint_{2^{-j}B} |\Pi_{2^{-j}\mathfrak{A}}u - (\Pi_{2^{-j}\mathfrak{A}}u)(x_0)|\,\mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned} \tag{3.3}$$

The term I is already suitable for later. We estimate

$$\begin{aligned}
\mathrm{II} &= 2\,\|\Pi_{2^{-j}B} - \Pi_{2^{-j}\mathfrak{A}}u\|_{L^\infty(2^{-j}B)} \\
&= 2\,\|\Pi_{2^{-j}\mathfrak{A}}(u - \Pi_{2^{-j}B}u)\|_{L^\infty(2^{-j}B)} \\
&\leq c\,\|\Pi_{2^{-j}\mathfrak{A}}(u - \Pi_{2^{-j}B}u)\|_{L^\infty(2^{-j}\mathfrak{A})} \\
&\leq c \fint_{2^{-j}\mathfrak{A}} |u - \Pi_{2^{-j}B}u| \,\mathrm{d}x \\
&\leq c \fint_{2^{-j}B} |u - \Pi_{2^{-j}B}u| \,\mathrm{d}x
\end{aligned}$$

using inverse estimates in the penultimate step. Let us estimate III. For notational brevity, put $p_j := \Pi_{2^{-j}\mathfrak{A}}u$. Next we employ a telescope sum argument to bound the term III by

$$\begin{aligned}
\mathrm{III} = \fint_{2^{-j}B} |p_j(x) - p_j(x_0)| \,\mathrm{d}x &\leq \fint_{2^{-j}B} |p_0(x) - p_0(x_0)| \,\mathrm{d}x \\
&\quad + \sum_{m=0}^{j-1} \fint_{2^{-j}B} |(p_{m+1} - p_m)(x) - (p_{m+1} - p_m)(x_0)| \,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

In conclusion, by Lemmas 2.5 and inverse estimates

$$\begin{aligned}
\mathrm{III}_m &:= \fint_{2^{-j}B} |(p_{m+1} - p_m)(x) - (p_{m+1} - p_m)(x_0)| \,\mathrm{d}x \\
&\leq c\, 2^{m-j} \fint_{2^{-m}B} |(p_{m+1} - p_m)(x) - (p_{m+1} - p_m)(x_0)| \,\mathrm{d}x \\
&\leq c\, 2^{m-j} \bigg(\fint_{2^{-m}B} |p_{m+1} - p_m| \,\mathrm{d}x + |(p_{m+1} - p_m)(x_0)| \bigg) \\
&\leq c\, 2^{m-j} \|p_{m+1} - p_m\|_{L^\infty(2^{-m}B)} \\
&\leq c\, 2^{m-j} \fint_{2^{-m}\mathfrak{A} \cap 2^{-m-1}\mathfrak{A}} |p_{m+1} - p_m| \,\mathrm{d}x.
\end{aligned}$$

Let us abbreviate $\mathfrak{A}_{m+1/2} := 2^{-m}\mathfrak{A} \cap 2^{-m-1}\mathfrak{A}$ and $p_{m+1/2} := \Pi_{2^{-m}\mathfrak{A} \cap 2^{-m-1}\mathfrak{A}}$. We estimate

$$\begin{aligned}
&\fint_{\mathfrak{A}_{m+1/2}} |p_{m+1} - p_m| \,\mathrm{d}x \\
&\quad\leq \fint_{\mathfrak{A}_{m+1/2}} |p_{m+1/2} - p_{m+1}| \,\mathrm{d}x + \fint_{\mathfrak{A}_{m+1/2}} |p_{m+1/2} - p_m| \,\mathrm{d}x \\
&\quad= c \fint_{\mathfrak{A}_{m+1/2}} |\Pi_{\mathfrak{A}_{m+1/2}}u - \Pi_{2^{-m-1}\mathfrak{A}}u| \,\mathrm{d}x + c \fint_{\mathfrak{A}_{m+1/2}} |\Pi_{\mathfrak{A}_{m+1/2}}u - \Pi_{2^{-m}\mathfrak{A}}u| \,\mathrm{d}x \\
&\quad\leq c \fint_{\mathfrak{A}_{m+1/2}} |\Pi_{\mathfrak{A}_{m+1/2}}(u - \Pi_{2^{-m-1}\mathfrak{A}}u)| \,\mathrm{d}x + c \fint_{\mathfrak{A}_{m+1/2}} |\Pi_{\mathfrak{A}_{m+1/2}}(u - \Pi_{2^{-m}\mathfrak{A}}u)| \,\mathrm{d}x \\
&\quad\leq c \fint_{\mathfrak{A}_{m+1/2}} |u - \Pi_{2^{-m-1}\mathfrak{A}}u| \,\mathrm{d}x + c \fint_{\mathfrak{A}_{m+1/2}} |u - \Pi_{2^{-m}\mathfrak{A}}u| \,\mathrm{d}x \\
&\quad\leq c \fint_{2^{-m-1}\mathfrak{A}} |u - \Pi_{2^{-m-1}\mathfrak{A}}u| \,\mathrm{d}x + c \fint_{2^{-m}\mathfrak{A}} |u - \Pi_{2^{-m}\mathfrak{A}}u| \,\mathrm{d}x.
\end{aligned}$$

On the other hand, since $\Pi_{\mathfrak{A}}$ is a projection, we have

$$\fint_B |p_0(x) - p_0(x_0)| \,\mathrm{d}x = \fint_B |(\Pi_{\mathfrak{A}}u)(x) - (\Pi_{\mathfrak{A}}u)(x_0)| \,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \fint_B \big|\big(\Pi_{\mathfrak{A}}(u-\langle u\rangle_{\mathfrak{A}}\big)(x)\big|\,\mathrm{d}x + \big|\big(\Pi_{\mathfrak{A}}(u-\langle u\rangle_{\mathfrak{A}})\big)(x_0)\big| \\
&\leq c \fint_{\mathfrak{A}} \big|\Pi_{\mathfrak{A}}(u-\langle u\rangle_{\mathfrak{A}})\big|\,\mathrm{d}x \\
&\leq c \fint_{\mathfrak{A}} \big|u-\langle u\rangle_{\mathfrak{A}}\big|\,\mathrm{d}x,
\end{aligned}
$$

where we have again used inverse estimates in the penultimate step and the L^1-stability of $\Pi_{\mathfrak{A}}$ (cf. (2.7)) in the last step.

We collect all estimates and get

$$
\begin{aligned}
\fint_{2^{-j}B} |u-\langle u\rangle_{2^{-j}B}|\,\mathrm{d}x \leq 2^{-j}c \fint_{\mathfrak{A}} &|u-\langle u\rangle_{\mathfrak{A}}|\,\mathrm{d}x \\
&+ c\fint_{2^{-j}B} |u-\Pi_{2^{-j}B}u|\,\mathrm{d}x \\
&+ c\sum_{m=0}^{j} 2^{m-j} \fint_{2^{-m}\mathfrak{A}} |u-\Pi_{2^{-m}\mathfrak{A}}u|\,\mathrm{d}x.
\end{aligned}
$$

This proves (3.1). The proof of (3.1) is analogous. Starting directly with $2^{-j}\mathfrak{A}$ instead of $2^{-j}B$ allows to avoid II and we obtain the better right-hand side. □

We now derive two useful consequences of Proposition 3.1.

Corollary 3.2. *Let $k<n$. Then there exists $c=c(\mathbb{A})>0$ such that for all $u\in L^1(B;V)$ there holds for each ball B with center x_0*

$$
\sum_{j=0}^{\infty} \fint_{2^{-j}B} |u-\langle u\rangle_{2^{-j}B}|\,\mathrm{d}x \leq c \fint_B |u-\langle u\rangle_B|\,\mathrm{d}x + c\,\mathcal{I}_k(\mathbb{A}u \llcorner B)(x_0).
$$

Proof. Let $\mathfrak{A} = B\setminus \frac{1}{4}\overline{B}$. We use Proposition 3.1 and sum over $j\in\mathbb{N}_0$ to get

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \fint_{2^{-j}B} |u-\langle u\rangle_{2^{-j}B}|\,\mathrm{d}x \\
&\qquad\leq c\fint_B |u-\langle u\rangle_B|\,\mathrm{d}x + \sum_{j=0}^{\infty}\fint_{2^{-j}B} |u-\Pi_{2^{-j}B}u|\,\mathrm{d}x \\
&\qquad\quad + c\sum_{j=0}^{\infty}\sum_{m=0}^{j} 2^{m-j}\fint_{2^{-m}\mathfrak{A}} |u-\Pi_{2^{-m}\mathfrak{A}}u|\,\mathrm{d}x =: \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}
$$

Term I is already in a convenient form, whereas III can be estimated via the Cauchy product by

$$
\begin{aligned}
\mathrm{III} &\leq c\sum_{m=0}^{\infty}\fint_{2^{-m}\mathfrak{A}} |u-\Pi_{2^{-m}\mathfrak{A}}u|\,\mathrm{d}x \\
&\leq c\sum_{m=0}^{\infty}(2^{-m}r)^{k-n}|\mathbb{A}u|(2^{-m}\mathfrak{A}) \\
&\leq c\int_B \frac{\mathrm{d}|\mathbb{A}u|(x)}{|x-x_0|^{n-k}} = c\,\mathcal{I}_k(\mathbb{A}u\llcorner B)(x_0).
\end{aligned}
$$

since $|x-x_0|\eqsim 2^{-m}r$ for any $x\in 2^{-m}\mathfrak{A}$ and the annuli $2^{-m}\mathfrak{A}$ have finite mutual overlap and all are contained in B. Turning to II, we have

$$
\sum_{j=0}^{\infty}\fint_{2^{-j}B}|u-\Pi_{2^{-j}B}u|\,\mathrm{d}x \leq c\sum_{j=0}^{\infty}(2^{-j}r)^{k-n}|\mathbb{A}u|(2^{-j}B)
$$

$$
\begin{aligned}
&\leq c\int_B \sum_{j=0}^{\infty}(2^{-j}r)^{k-n}\mathbb{1}_{2^{-j}B}\,\mathrm{d}|\mathbb{A}u| \\
&\leq c\int_B |x-x_0|^{k-n}\,\mathrm{d}|\mathbb{A}u| \\
&= \mathcal{I}_k(\mathbb{A}u\llcorner B)(x_0).
\end{aligned}
$$

Gathering estimates, the proof is complete. □

3.2. **Proof of Theorem 1.1.** After the preparations from the preceding subsection, we now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1 (a). Let $u\in \mathrm{BV}^{\mathbb{A}}_{\mathrm{loc}}(\mathbb{R}^n)$, $x_0\in\mathbb{R}^n$ and $r>0$ be such that $\mathcal{I}_k(\mathbb{A}u\llcorner B(x_0,r))(x_0)<\infty$. Then, by Corollary 3.2, we have

$$
\begin{aligned}
|\langle u\rangle_{2^{-j}B}-\langle u\rangle_{2^{-l}B}| &\leq \sum_{i=l}^{j}|\langle u\rangle_{2^{-i}B}-\langle u\rangle_{2^{-i-1}B}| \\
&\leq 2^n\sum_{i=l}^{j}\fint_{2^{-i}B}|u-\langle u\rangle_{2^{-i}B}|\,\mathrm{d}x\to 0
\end{aligned}
\tag{3.5}
$$

as $j,l\to\infty$. Therefore, $\langle u\rangle_{2^{-i}B}\to q$ as $i\to\infty$ for some $q\in V$ and so, by Corollary 3.2,

$$
\fint_{2^{-i}B}|u-q|\,\mathrm{d}x\leq \fint_{2^{-i}B}|u-\langle u\rangle_{2^{-i}B}|\,\mathrm{d}x+|\langle u\rangle_{2^{-i}B}-q|\to 0.
$$

Since it suffices to consider balls $2^{-i}B$, the proof is complete. □

Proof of Theorem 1.1 (b). Recall that now $k=n$. Let $x_0\in\mathbb{R}^n$; our first objective is to prove that x_0 is a Lebesgue point for $u\in\mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$. Let $B=B(x_0,r)$ and $\mathfrak{A}:=B\setminus\frac14\overline{B}$. Then it follows from Proposition 3.1 and Corollary 2.3 (applied with $\lambda=0$ and $\lambda=\frac14$) that

$$
\begin{aligned}
\fint_{2^{-j}B}|u-\langle u\rangle_{2^{-j}B}|\,\mathrm{d}x &\leq 2^{-j}c\fint_{\mathfrak{A}}|u-\langle u\rangle_{\mathfrak{A}}|\,\mathrm{d}x \\
&\qquad + c\,|\mathbb{A}u|(2^{-j}B\setminus\{x_0\})+c\sum_{m=0}^{j}2^{m-j}|\mathbb{A}u|(2^{-m}\mathfrak{A}) \\
&\leq 2^{-j}c\fint_B|u-\langle u\rangle_B|\,\mathrm{d}x+c|\mathbb{A}u|(B\setminus\{x_0\}).
\end{aligned}
$$

Thus, if follows that for $0<s<\frac{r}{2}$ we have

$$
\fint_{B(x_0,s)}|u-\langle u\rangle_{B(x_0,s)}|\,\mathrm{d}x\leq c\frac{s}{r}\fint_{B(x_0,r)}|u-\langle u\rangle_{\mathfrak{A}}|\,\mathrm{d}x+c|\mathbb{A}u|(B(x_0,r)\setminus\{x_0\}).
$$

Let $\varepsilon>0$ be arbitrary. Since $B(x_0,r)\setminus\{x_0\}\to\emptyset$ for $r\to0$ and $|\mathbb{A}u|$ is a Radon measure, hence outer regular, we find $r>0$ such that $|\mathbb{A}u|(B(x_0,r)\setminus\{x_0\})<\frac{\varepsilon}{2}$. Now, we can choose s so small that also the integral on the right hand side becomes small than $\frac{\varepsilon}{2}$. Since $\varepsilon>0$ was arbitrary this proves that

$$
\lim_{s\searrow0}\fint_{B(x_0,s)}|u-\langle u\rangle_{B(x_0,s)}|\,\mathrm{d}x=0.
\tag{3.6}
$$

Recall that $B=B(x_0,r)$ and put $\mathfrak{A}:=B\setminus\frac14\overline{B}$. Then (3.6) implies that

$$
\lim_{j\to\infty}|\langle u\rangle_{2^{-j}B}-\langle u\rangle_{2^{-j}\mathfrak{A}}|\leq c\lim_{j\to\infty}\fint_{2^{-j}B}|u-\langle u\rangle_{2^{-j}B}|\,\mathrm{d}x=0.
\tag{3.7}
$$

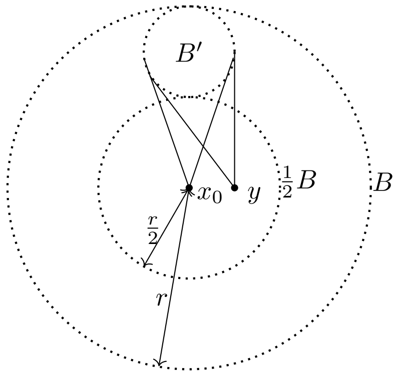

FIGURE 1. The geometric situation in the proof of Theorem 1.1(b).

Now, by Proposition 3.1 and Corollary 2.3 (with $\lambda = 0$ and $\lambda = \frac{1}{4}$) and $k = n$ we have

$$\begin{aligned}
&|\langle u\rangle_{2^{-j}\mathfrak{A}} - \langle u\rangle_{2^{-j-1}\mathfrak{A}}| \\
&\quad\leq c \fint_{2^{-j}\mathfrak{A}} |u - \langle u\rangle_{2^{-j}\mathfrak{A}}| \,\mathrm{d}x \\
&\quad\leq c\, 2^{-j} \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}| \,\mathrm{d}x + c \sum_{m=0}^{j} 2^{m-j} \fint_{2^{-m}\mathfrak{A}} |u - \Pi_{2^{-m}\mathfrak{A}} u| \,\mathrm{d}x \\
&\quad\leq c\, 2^{-j} \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}| \,\mathrm{d}{+}c \sum_{m=0}^{j} 2^{m-j} |\mathbb{A}u|(2^{-m}\mathfrak{A}).
\end{aligned}$$

Summing over $j \geq 0$ we obtain

$$\begin{aligned}
\sum_{j\geq 0} |\langle u\rangle_{2^{-j}\mathfrak{A}} - \langle u\rangle_{2^{-j-1}\mathfrak{A}}| &\leq c \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}| \,\mathrm{d}x + c \sum_{j\geq 0}\sum_{m=0}^{j} 2^{m-j} |\mathbb{A}u|(2^{-m}\mathfrak{A}) \\
&\leq c \fint_{\mathfrak{A}} |u - \langle u\rangle_{\mathfrak{A}}| \,\mathrm{d}x + c\,|\mathbb{A}u|(B \setminus \{x_0\}).
\end{aligned} \tag{3.8}$$

This proves that $u_0 := \lim_{j\to\infty} \langle u\rangle_{2^{-j}\mathfrak{A}}$ exists. Due to (3.7) we see that also $u_0 = \lim_{j\to\infty} \langle u\rangle_{2^{-j}B}$. Thus, it follows by (3.6) that

$$\lim_{j\to\infty} \fint_{2^{-j}B} |u - u_0| \,\mathrm{d}x \leq \limsup_{j\to\infty} \fint_{2^{-j}B} |u - \langle u\rangle_{2^{-j}B}| \,\mathrm{d}x + \limsup_{j\to\infty} |\langle u\rangle_{2^{-j}B} - u_0| = 0.$$

This proves that x_0 is a Lebesgue point. Since x_0 was arbitrary, we see that all points of u are Lebesgue points. In the following we choose u to be the unique representative, which coincides with Lebesgue point limits.

Again let B be ball with center x_0 with radius $r > 0$. Now, let $y \in \frac{1}{2}B$ be fixed. Then we can choose a ball $B' \subset B \setminus \frac{1}{2}\overline{B}$ such that the sets

$$\begin{aligned}
C_{x_0} &:= \operatorname{conv\,hull}(\overline{B'} \cup \{x_0\}), \\
C_y &:= \operatorname{conv\,hull}(\overline{B'} \cup \{y\}),
\end{aligned}$$

satisfy $y \notin C_{x_0}$ and $x \notin C_y$. This geometric constellation is depicted in Figure 1. Now, we use the representation formula (2.1) with $B_\Omega = B'$ to get

$$\begin{aligned}
|u(x_0) - (\mathbb{P}^m_{B'} u)(x_0)| &\leq c\,|\mathbb{A}u|(C_{x_0}), \\
|u(y) - (\mathbb{P}^m_{B'} u)(y)| &\leq c\,|\mathbb{A}u|(C_y).
\end{aligned} \tag{3.9}$$

This can be improved to

$$
\begin{aligned}
|u(x_0) - (\mathbb{P}^m_{B'}u)(x_0)| &\le c\,|\mathbb{A}u|(C_{x_0} \setminus \{x_0\}),\\
|u(y) - (\mathbb{P}^m_{B'}u)(y)| &\le c\,|\mathbb{A}u|(C_y \setminus \{y\}).
\end{aligned}
\tag{3.10}
$$

as follows: It suffices to prove the first estimate. Denote x_0' the center of B' and put $x_j := (1 - \frac{1}{j})x_0 + \frac{1}{j}x_0'$ and choose $\theta \in (0,1)$ so small that $B(x_j, 2\theta|x_j - x_0|) \subset C_{x_0} \setminus \{x_0\}$. Taking the average of the representation formula (2.1) for every $x \in B(x_j, 2\theta|x_j - x_0|)$ we obtain

$$
\begin{aligned}
\big|\langle u\rangle_{B(x_j,2\theta|x_j-x_0|)} - \langle \mathbb{P}^m_{B'}u\rangle_{B(x_j,2\theta|x_j-x_0|)}\big| &\le c\,|\mathbb{A}u|(\operatorname{conv}\operatorname{hull}(\overline{B'} \cup \{x_0\}))\\
&\le c|\mathbb{A}u|(C_{x_0} \setminus \{x_0\})
\end{aligned}
$$

using also that $\operatorname{conv}\operatorname{hull}(\overline{B'} \cup \{x_0\}) \subset C_{x_0} \setminus \{x_0\}$ for every $x \in B(x_j, 2\theta|x_j - x_0|)$. Now, (3.10) follows by passing with $j \to \infty$ and using that x_0 is a Lebesgue point and that $\mathbb{P}^m_{B'}u$ is continuous.

Using (3.10) we get

$$
\begin{aligned}
|u(x_0) - u(y)| &\le |u(x_0) - (\mathbb{P}^m_{B'}u)(x_0)| + |u(y) - (\mathbb{P}^m_{B'}u)(x_0)|\\
&\quad + |(\mathbb{P}^m_{B'}u)(x_0)) - (\mathbb{P}^m_{B'}u)(y)|\\
&\le c\,|\mathbb{A}u|(C_{x_0} \setminus \{x_0\}) + c\,|\mathbb{A}u|(C_y \setminus \{y\}) + |(\mathbb{P}^m_{B'}u)(x_0)) - (\mathbb{P}^m_{B'}u)(y)|\\
&\le c\,|\mathbb{A}u|(B \setminus \{x_0\}) + |(\mathbb{P}^m_{B'}u)(y)) - (\mathbb{P}^m_{B'}u)(x_0)|.
\end{aligned}
$$

We further estimate

$$
\begin{aligned}
|(\mathbb{P}^m_{B'}u)(y)) - (\mathbb{P}^m_{B'}u)(x_0)| &\le |x_0 - y|\,\|\nabla(\mathbb{P}^m_{B'}u)\|_{L^\infty(B)}\\
&\le c\,\frac{|x_0 - y|}{r} \fint_{B'} |\mathbb{P}^m_{B'}(u - \langle u\rangle_B)|\,\mathrm{d}x\\
&\le c\,\frac{|x_0 - y|}{r} \fint_{B'} |u - \langle u\rangle_B|\,\mathrm{d}x\\
&\le c\,\frac{|x_0 - y|}{r} \fint_{B} |u - \langle u\rangle_B|\,\mathrm{d}x,
\end{aligned}
$$

where we have used inverse estimates for polynomials and the L^1-stability of the averaged Taylor polynomial. Overall, we obtain

$$
|u(x_0) - u(y)| \le c\,|\mathbb{A}u|(B \setminus \{x_0\}) + c\,\frac{|x_0 - y|}{r} \fint_B |u - \langle u\rangle_B|\,dx. \tag{3.11}
$$

This proves that u is continuous at x_0 and also (1.7). Indeed, for given $\varepsilon > 0$ we choose $r > 0$ such that $|\mathbb{A}u|(B \setminus \{x_0\}) < \varepsilon$. Then, taking the constant $c > 0$ from (3.11), we choose $0 < \delta < r$ so small such that $|x_0 - y| < \delta$ implies $c|x_0 - y|r^{-1} \fint_B |u|\,\mathrm{d}z < \varepsilon$. This yields $|u(x_0) - u(y)| < c\varepsilon$ for all $y \in \mathbb{R}^n$ with $|x_0 - y| < \delta$, and so u is continuous. The proof is complete. □

Proof of Theorem 1.1 (c). Recall that now $\mathbb{A}$ is a $\mathbb{C}$-elliptic differential operator of order $k > n$. Then we can proceed exactly as in the case (b) but with $\nabla^{k-n}u$ instead of u. In the applications of the Poincaré type estimates on the annuli (see Corollary 2.3) we choose $\ell = k - n$. With the same arguments as in the proof of Theorem 1.1(b) we find that every point of $\nabla^{n-k}u$ is a Lebesgue point and that for all $x \in \mathbb{R}^n$ and $r > 0$

$$
\begin{aligned}
|(\nabla^{k-n}u)(x_0) - (\nabla^{k-n}u)(y)| &\le c\,|\mathbb{A}u|(B \setminus \{x_0\})\\
&\quad + c\,\frac{|x_0 - y|}{r} \fint_B |\nabla^{k-n}u - \langle \nabla^{k-n}u\rangle_B|\,\mathrm{d}x.
\end{aligned}
\tag{3.12}
$$

for all $y \in \mathbb{R}^n$ with $|x-y| < \frac{1}{2}r$. This implies again that $\nabla^{k-n}u$ is continuous, so $u \in C^{k-n}$. This proves the claim. $\square$

Remark 3.3. *Note that from the representation formula, see also* (3.9)*, we immediately obtain for $k \geq n$ the local L^∞-bound*

$$\|\nabla^{k-n}u\|_{L^\infty(B)} \leq c \fint_B |\nabla^{k-n}u|\,dx + c\,|\mathbb{A}u|(B). \tag{3.13}$$

This also implies (with $B \to \mathbb{R}^n$) that for $k \geq n$

$$\|\nabla^{k-n}u\|_{L^\infty(\mathbb{R}^n)} \leq c\,|\mathbb{A}u|(\mathbb{R}^n). \tag{3.14}$$

Moreover, the extension operator from [10] *allows us to apply these estimates and Theorem 1.1 to bounded Lipschitz domains. In particular, we obtain for $k \geq n$ the embedding $\mathrm{BV}^{\mathbb{A}}(\Omega) \hookrightarrow C^{k-n}(\overline{\Omega}; V)$.*

Remark 3.4 (Singletons)**.** *The step from* (3.9) *to* (3.10) *can also be obtained by a different argument, namely that $|\mathbb{A}u|(\{x_0\}) = 0$ for each x_0, i.e. $\mathbb{A}u$ cannot charge singletons.*

One possibility to prove this is that every $\mathbb{C}$-elliptic operator is elliptic and cancelling (cf. [10]*) and thus by* Van Schaftingen [24, Prop. 2.1]*, $|\mathbb{A}u|$ cannot charge singletons.*

Let us present an alternative proof based on the trace theorem of [5]*, for simplicity stated for first order operators. Let Σ be a Lipschitz hypersurfaces passing through the points x_0. Then by the gluing theorem* [5, Proposition 4.12]

$$\mathbb{A}u \llcorner \Sigma = (u_\Sigma^+ - u_\Sigma^-) \otimes_{\mathbb{A}} \nu \mathscr{H}^{n-1} \llcorner \Sigma, \tag{3.15}$$

where ν is the unit normal to Σ and u_Σ^+, u_Σ^- are the left- or right-sided traces along Σ, respectively, which exist in $L^1(\Sigma; V)$ by [5, Thm. 1.1]*. This implies*

$$\mathbb{A}u \llcorner \Sigma \ll \mathscr{H}^{n-1} \llcorner \Sigma. \tag{3.16}$$

Thus $\mathbb{A}u$ cannot charge any $\mathscr{H}^{n-1}$-nullset contained in Σ. Thus, in particular, $|\mathbb{A}u|(\{x_0\}) = 0$. Let us note that, based on the proof of [5, Thm. 4.18]*, for elliptic first order operators $\mathbb{A}$,* (3.16) *for all $u \in \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ and Lipschitz hypersurfaces Σ is in fact equivalent to $\mathbb{C}$-ellipticity.*

Remark 3.5. *The representability* (3.15) *could also be used to describe points in the jump set J_u on hypersurfaces. However, working from Theorem 1.1(a), it is not immediately clear how the Riesz potential criterion should yield $|\mathbb{A}u|(S_u \setminus J_u) = 0$, which would be required for a proper structure theory for $\mathrm{BV}^{\mathbb{A}}$-maps. We intend to tackle this question in the future.*

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