L^1-Estimates and Function Spaces Research Overview

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1 Introduction

Let $\mathbb{A} = \sum_{|\alpha|=k} \mathbb{A}_\alpha \partial^\alpha$ be a constant-coefficient, linear, k-th order homogeneous differential operator on $\mathbb{R}^n$ between the finite dimensional real vector spaces V and W, where $\mathbb{A}_\alpha \colon V \to W$ are fixed linear maps. Building on the foundational work of Calderón & Zygmund and Mihlin, it is a by now well-established fact that an estimate of the form

$$\|D^k\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;\mathrm{SLin}^k(\mathbb{R}^n;V))} \leq c\|\mathbb{A}\varphi\|_{\mathrm{L}^p(\mathbb{R}^n;W)} \qquad \text{for all } \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;V) \tag{1.1}$$

holds for $1 < p < \infty$ if and only if $\mathbb{A}$ is *elliptic*. The latter asserts that for each $\xi \in \mathbb{R}^n \setminus \{0\}$, the symbol map $\mathbb{A}[\xi] \colon V \to W$ is injective. Inequality (1.1) then is a consequence of the fact that the maps $\Phi \colon \mathbb{A}\varphi \mapsto \partial^\beta\varphi$ given by $\Phi(\mathbb{A}u) = \mathscr{F}^{-1}(\xi^\beta(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}[\xi]\widehat{\mathbb{A}u})$ is Fourier multiplication operator homogeneous of degree zero. This theorem in general neither extends to $p = 1$ or $p = \infty$, a negative result which is often referred to as *Ornstein's Non-Inequality*. In this vein, given an open domain $\Omega \subset \mathbb{R}^n$, the foremost question which we are after is *which properties of φ can be deduced when $\mathbb{A}[D]\varphi$ belongs to $\mathrm{L}^1(\Omega;W)$ or $\mathscr{M}(\Omega;W)$, the W-valued finite Radon measures on Ω.*

This question can be recasted as asking for embeddings of function spaces

$$\begin{aligned} \mathrm{W}^{\mathbb{A},p}(\Omega) &:= \big\{u \in \mathrm{L}^p(\Omega;V) \colon\ \mathbb{A}u \in \mathrm{L}^p(\Omega;W)\big\}, \\ \mathrm{BV}^{\mathbb{A}}(\Omega) &:= \big\{u \in \mathrm{L}^1(\Omega;V) \colon\ \mathbb{A}u \in \mathscr{M}(\Omega;W)\big\}. \end{aligned} \tag{1.2}$$

These spaces allow to come up with a unifying framework within which, e.g., functions of bounded variation (where $V = \mathbb{R}^n, W = \mathbb{R}^N$ and $\mathbb{A}u = Du$) and functions of bounded deformation (where $V = W = \mathbb{R}^n$ and $\mathbb{A}u = \varepsilon(u) = \frac{1}{2}(Du + Du^\top)$) can be studied. However, the theory of what is called $\mathbb{A}$-*weakly differentiable functions* is much richer, and the research program laid down in this exposé aims to explain what has been achieved so far and which are the future directions.

2 Embedding and Trace Theory

Not from a function space but pure inequality perspective, J. Van Schaftingen [VSc08, VSc13] established that an inequality of the form $\|\varphi\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;V)} \leq c\|\mathbb{A}\varphi\|_{\mathrm{L}^1(\mathbb{R}^n;W)}$ holds for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n;V)$ if and only if $\mathbb{A}[D]$ is EC, i.e., elliptic *and cancelling*; this systematised previous work by Bourgain & Brezis [BB04,BB07]. The latter means that $\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \mathbb{A}[\xi](V) = \{0\}$ and hence, heuristically, the symbol map must be sufficiently spread. In particular, the Sobolev estimate does not follow from ellipticity in itself. The same condition also yields inequalities that yield suitable Besov- or Triebel-Lizorkin-norms of φ against the L^1-norm of $\mathbb{A}\varphi$. Such inequalities can be interpreted as embeddings for the homogeneous variants of $\mathrm{BV}^{\mathbb{A}}$-spaces. On an informal level, Ornstein's Non-Inequality is essentially not visible on lower order derivatives or the functions themselves for a wealth of operators $\mathbb{A}$.

In a similar spirit, in collaboration with D. Breit (Edinburgh) and L. Diening (Bielefeld) [1] we studied boundary trace inequalities – where $\Omega \subset \mathbb{R}^n$ is an open and bounded

Lipschitz domain – for first order operators $\mathbb{A}$

$$\|\varphi\|_{\mathrm{L}^1(\partial\Omega;V)} \leq c\big(\|\varphi\|_{\mathrm{L}^1(\Omega;V)} + |\mathbb{A}\varphi|(\Omega)\big) \qquad \text{for all } \varphi \in \mathrm{C}(\overline{\Omega};V)\cap \mathrm{BV}^{\mathbb{A}}(\Omega). \tag{2.1}$$

Due to the possibly complicated structure of $\mathbb{A}$, this issue cannot be approached easily by the fundamental theorem of calculus. As an outcome, we could isolate the algebraic property of $\mathbb{C}$-ellipticity on the operator $\mathbb{A}$ to be necessary and sufficient for (2.1) to hold. $\mathbb{C}$-ellipticity means that for each $\xi \in \mathbb{C}^n \setminus \{0\}$, the complixified symbol map $\mathbb{A}[\xi]\colon V + \mathrm{i}V \to W + \mathrm{i}W$ remains injective. This condition again is strictly stronger than mere ellipticity, and most notably implies that the nullspace of $\mathbb{A}$ is *finite dimensional*. This allowed us to find a novel approach to constructing traces by employing a careful local projection procedure on the nullspaces of $\mathbb{A}$ on small balls approaching the boundary $\partial\Omega$. The condition of $\mathbb{C}$-ellipticity even turns out necessary and sufficient for (2.1) in the framework of higher order differential operators [5].

In a next step, it is natural to ask whether VAN SCHAFTINGEN's full space, homogeneous inequalities hold on domains subject to non-zero boundary data. Moreover, it is desirable to study how EC and $\mathbb{C}$-elliptic operators are related. Jointly with B. RAITA (Warwick) [2] we established in a slightly larger framework that inequalities of the form

$$\|\varphi\|_{F^s_{p,q}(\Omega;V)} \leq c\big(\|\varphi\|_{\mathrm{L}^1(\Omega;V)} + \|\mathbb{A}\varphi\|_{\mathrm{L}^1(\Omega;W)}\big) \qquad \text{for all } \varphi \in \mathrm{C}(\overline{\Omega};V)\cap \mathrm{BV}^{\mathbb{A}}(\Omega) \tag{2.2}$$

for essentially the same possible choices of s, p, q as in the full gradient case persist for $\mathbb{C}$-elliptic operators $\mathbb{A}[D]$. Suprisingly, $\mathbb{C}$-ellipticity turns out much stronger than EC [2], which also sheds a new light on the underlying mechanisms for Sobolev- and trace embeddings to work for the by now classical cases of the gradient and symmetric gradient operators.

In collaboration with B. RAITA (Warwick) and J. VAN SCHAFTINGEN (Louvain) [5] we currently study the analogues of ADAMS' trace inequalities [Ad71]. Roughly speaking, if $\Sigma \subset \mathbb{R}^n$ is a sufficiently regular subset such that $\mathscr{H}^{n-s} \llcorner \Sigma$ is locally finite, then EC *still suffices to assign traces* of $u \in \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n)$ along Σ *provided* $0 \leq s < 1$. In this work we also introduce new notions and refinements of the EC and $\mathbb{C}$-ellipticity conditions which shall be instrumental when studying more general boundary estimates. In conclusion, as a heuristic metaprinciple, the L^1-trace embeddings on Lipschitz submanifolds *of codimension one refer absolute endpoint estimates*, one thus has:

Absolute endpoint estimates are usually only obtainable subject to $\mathbb{C}$-ellipticity.

Current research [4] also concerns the sharp trace theorem for higher order operators due to USPENSKIĬ (i.e., $\mathrm{W}^{2,1}(\Omega) \hookrightarrow B^1_{1,1}(\partial\Omega)$) and accompanying $\mathbb{A}$-free oscillation characterisations of Besov spaces.

3 Fine and Differentiability Properties

If $u \in \mathrm{BV}(\mathbb{R}^n)$, a rich structure theory is available. For instance, the singular part $D^s u \perp \mathscr{L}^n$ in the Lebesgue-Radon-Nikodým decomposition of $Du = D^a u + D^s u$, can be split into $D^s u = D^c u + D^j u := D^s u \llcorner S_u^c + D^s u \llcorner J_u$. Here, S_u are the Lebesgue discontinuity points (for $\mathscr{L}^n$) and J_u are the so-called *jump points*. This result also involves the fact that $|D^s u|(S_u \setminus J_u) = 0$. See AMBROSIO et al. [AFP00] for more details.

For general elliptic operators, an analogous splitting is not possible; see, for instance, $\mathbb{A}[D] = \varepsilon^{\mathrm{tf}}$ the trace-free symmetric gradient in two dimensions and note that $\mathbb{A}^s[D]u \llcorner S_u^c \neq \mathbb{A}^s[D]u \llcorner J_u$ in general. The understanding of the fine properties of $\mathrm{BV}^{\mathbb{A}}$-maps are subject of current interest, but only little is known at present. However, first steps have been taken by B. RAITA (Warwick) and myself in proving a borderline case left open by ALBERTI et al. [ABC14]. More precisely, based on [2], in [3] we showed that for $\mathbb{C}$-elliptic $\mathbb{A}[D]$, $\mathrm{BV}^{\mathbb{A}}$-maps are $\mathrm{L}^{\frac{n}{n-1}}$-differentiable $\mathscr{L}^n$-a.e.. The approach once again crucially involved scaling properties and inverse estimates for polynomials of a fixed degree. In future contributions, we hope to obtain a satisfactory – partially analogous as for BV – theory for $\mathbb{C}$-elliptic or EC operators.

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