EMBEDDINGS FOR $\mathbb{A}$–WEAKLY DIFFERENTIABLE FUNCTIONS ON DOMAINS

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Abstract. We prove that the inhomogeneous estimate of vector fields on balls in $\mathbb{R}^n$

$$\left(\int_B |D^{k-1}u|^{n/(n-1)}\mathrm{d}x\right)^{(n-1)/n} \leqslant c\left(\int_B |\mathbb{A}u| + |u|\mathrm{d}x\right) \text{ for all } u \in \mathrm{C}^\infty(\bar{B}, \mathbb{R}^N)$$

holds if and only if the linear, constant coefficient differential operator $\mathbb{A}$ of order k has finite dimensional null-space (FDN). This generalizes the Gagliardo-Nirenberg-Sobolev inequality on domains and provides the local version of the analogous homogeneous embedding in full-space

$$\left(\int_{\mathbb{R}^n} |D^{k-1}u|^{n/(n-1)}\mathrm{d}x\right)^{(n-1)/n} \leqslant c\int_{\mathbb{R}^n} |\mathbb{A}u|\mathrm{d}x \qquad \text{for all } u \in \mathrm{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^N),$$

proved by Van Schaftingen precisely for elliptic and cancelling (EC) operators, building on fundamental L^1-estimates from the works of Bourgain and Brezis. We prove that FDN strictly implies EC and discuss the contrast between homogeneous and inhomogeneous estimates on both algebraic and analytic level.

1. Introduction

1.1. **L^1-estimates.** A known principle in harmonic analysis is that strong–type L^1–estimates are notoriously delicate to obtain. For example, singular integrals and Riesz potentials are only bounded from L^1 into a weak–type space, which contrasts the case of L^p–spaces, $p > 1$. To note that these L^1–estimates of weak–type are sharp, one simply tests the inequalities with an approximation of the identity.

Historically, this discrepancy can be observed already from the original proof of the Sobolev inequality for $1 < p < n$ and $p^* = \frac{np}{n-p}$,

$$\|u\|_{\mathrm{L}^{p^*}(\mathbb{R}^n)} \leqslant c\|Du\|_{\mathrm{L}^p(\mathbb{R}^n)} \tag{1.1}$$

for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n)$. The technique of proof in [41] resembles proving boundedness of the Riesz potential I_1 between L^p and $\mathrm{L}^{np/(n-p)}$, but it is in no way adaptable to the $p = 1$ case. It was much later that Gagliardo [24] and Nirenberg [36] independently showed with new methods that (1.1) holds also for $p = 1$, in particular showing that the specific vectorial structure of the gradient operator allows for a strong–type estimate, despite unboundedness of I_1 between L^1 and L^{1^*}.

Coupled with the fact attributable to Calderón and Zygmund [16] that

$$\|D^k u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c\|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \tag{1.2}$$

for k–th order elliptic operators $\mathbb{A}$ and $1 < p < \infty$, it seems plausible that the vectorial structure of $\mathbb{A}$ may also compensate for unboundedness of singular integrals on L^1. This fact was disproved by Ornstein in [37] (see also [31]), where it is shown that the estimate (1.2) holds for $p = 1$ only if $|D^k u| \leqslant c|\mathbb{A}u|$ pointwisely for all $u \in \mathrm{C}_c^\infty$.

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However, it remained possible that strong–type L^1–estimates for lower order derivatives can be deduced. Indeed, it was proved by STRAUSS in [44] that (1.1) holds for $p=1$ when the L^1–norm of Du is replaced by the weaker quantity $\|\mathcal{E}u\|_{\mathrm{L}^1}$. Here $\mathcal{E}u = \frac{1}{2}(Du+(Du)^t)$ denotes the symmetrized gradient of u. More recently, it was proved by BOURGAIN and BREZIS in [5, 7] that, for Poisson's equation in $\mathbb{R}^n$, $n\geq 2$,

$$\Delta u = f,$$

the surprising strong L^1–estimate

$$\|Du\|_{\mathrm{L}^{1^*}} \leqslant c\|f\|_{\mathrm{L}^1}$$

holds provided that $f\in \mathrm{C}^\infty_c$ is divergence–free. This and substantial contributions in [8, 9, 10, 4, 6, 5, 7, 48, 49, 50, 51] lead to the remarkable characterization by VAN SCHAFTINGEN [52] of all k–homogeneous linear differential operators $\mathbb{A}$ such that

$$\|D^{k-1}u\|_{\mathrm{L}^{1^*}(\mathbb{R}^n)} \leqslant c\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n)} \tag{1.3}$$

for all $u\in \mathrm{C}^\infty_c$. The class of operators $\mathbb{A}$ for which (1.3) holds is that of *elliptic* and *cancelling* operators (EC). Both these assumptions are defined in terms of the symbol map of the operator $\mathbb{A}$, the definition of which we now recall. We will represent k–homogeneous linear differential operators with constant coefficients on $\mathbb{R}^n$ from V to W as

$$\mathbb{A}u = \sum_{|\alpha|=k} A_\alpha \partial^\alpha u, \qquad u\colon \mathbb{R}^n\to V, \tag{1.4}$$

where $A_\alpha\in \mathscr{L}(V,W)$ are fixed linear mappings between two finite dimensional normed real vector spaces V and W. The symbol map is defined as

$$\mathbb{A}[\cdot]\colon \mathbb{R}^n\to\mathscr{L}(V,W),\quad \mathbb{A}[\xi]v = \sum_{|\alpha|=k}\xi^\alpha A_\alpha v,$$

defined for $\xi\in\mathbb{R}^n$, $v\in V$. Algebraically, (overdetermined) *ellipticity* is defined by injectivity of the symbol map $\mathbb{A}[\xi]$ for all non–zero $\xi\in\mathbb{R}^n$, whereas *cancellation*, introduced in [52, Def. 1.2], is defined by

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \operatorname{im}\mathbb{A}[\xi] = \{0\}. \tag{1.5}$$

We will use the short–hand EC for operators that are elliptic and cancelling. Analytically, ellipticity is equivalent to the classical estimate (1.2). Surprisingly and interestingly, cancellation is equivalent to non–admissibility for (1.3) of approximations of the identity, in the sense that if $\mathbb{A}u_\varepsilon = \varphi_\varepsilon w$ for all standard mollifiers $\varphi_\varepsilon\in \mathrm{C}^\infty_c(\mathbb{R}^n)$ and $w\in W$, then $w\in\operatorname{im}\mathbb{A}[\xi]$ for all $\xi\neq 0$.

1.2. **L^1-estimates on domains.** An overarching overview of these and other recent developments on L^1–estimates can be found in [53], where it is also asked in Open Problem 3 whether, under suitable complementing boundary conditions, one can develop global strong–type estimates on domains. It is implicitly conjectured that estimates on smooth domains $\Omega\subset\mathbb{R}^n$ such as

$$\|D^{k-1}u\|_{\mathrm{L}^{1^*}(\Omega)} \leqslant \|\mathbb{A}u\|_{\mathrm{L}^1(\Omega)} + \|u\|_{\mathrm{L}^1(\Omega)},$$

provided that $\mathbb{A}$ is EC and $u\in \mathrm{C}^\infty(\bar\Omega,V)$ satisfy $\mathbb{B}_j u = 0$ on $\partial\Omega$, where $\mathbb{B}_j$ is a (finite collection of) linear differential operator(s) defined on $\partial\Omega$ that satisfy the Lopatinskiĭ–Shapiro Complementing Conditions. Such a result would provide a reasonable analogue of the results in [32, 1, 2, 26] to the case $p=1$, in spite of Ornstein's Non–Inequality.

The aim of this paper is to confirm this expectation in the case when $\mathbb{B}_j\equiv 0$ ("no boundary condition") and Ω is a ball (whereas VAN SCHAFTINGEN's result [52, Thm. 1.3] essentially deals with the antipodal case when $\mathbb{B}_j = \partial^j_\nu$, $j=0\dots k-1$, i.e., "all boundary conditions"). We emphasize that in the present situation *the geometry of $\partial\Omega$ is not the foremost problem*, as is the extendibility of functions $u\colon\Omega\to V$ to some $v\colon\mathbb{R}^n\to V$ while ensuring that $\mathbb{A}v \in \mathrm{L}^1(\mathbb{R}^n, V)$ boundedly. In fact, as we shall see below, this property even fails for a wealth of elliptic and cancelling operators. However, as the reader will note at ease, the extension procedure as outlined below works for a substantially larger class than that of Lipschitz domains, e.g., for those covered in [28, 18].

The complementing conditions for smooth domains for an elliptic operator $\mathbb{A}$ and identically zero boundary conditions [27, Def. 20.1.1] can be rephrased as

$$\mathbb{A}[\xi] \in \mathscr{L}(V + \mathrm{i}\,V, W + \mathrm{i}\,W) \qquad \text{is injective for all } \xi \in \mathbb{C}^n \setminus \{0\}. \tag{1.6}$$

This condition, referred to as $\mathbb{C}$–*ellipticity* in [12], and attributable to Aronszajn [3] (at least in the case of scalar–valued maps), was introduced to characterize operators $\mathbb{A}$ such that the local variant of (1.2) holds, i.e.,

$$\|D^k u\|_{\mathrm{L}^p(B)} \leqslant c\left(\|\mathbb{A}u\|_{\mathrm{L}^p(B)} + \|u\|_{\mathrm{L}^p(B)}\right) \tag{1.7}$$

holds for $u \in \mathrm{C}^\infty(\bar{B}, V)$ (here $1 < p < \infty$); see also [39, 40]. By Ornstein's Non–Inequality, no such estimate is possible for $p = 1$, but, inspired by Van Schaftingen's Theorem and Open Problem, we will prove that $\mathbb{C}$–ellipticity of $\mathbb{A}$ is equivalent to the estimate

$$\|D^{k-1} u\|_{\mathrm{L}^{n/(n-1)}(B)} \leqslant c\left(\|\mathbb{A}u\|_{\mathrm{L}^1(B)} + \|u\|_{\mathrm{L}^1(B)}\right) \tag{1.8}$$

for $u \in \mathrm{C}_c^\infty(\bar{B}, V)$. In particular, we recover the Gagliardo–Nirenberg–Sobolev inequality on domains and the Korn–Sobolev inequality [43, Prop. 1.2], due to Strang and Temam.

To formally state our results, we prefer to use the functional framework of $\mathbb{A}$–weakly differentiable functions and define, in the spirit of [12], the space $\mathrm{W}^{\mathbb{A},1}(B)$ as the space of $u \in \mathrm{L}^1(B, V)$ such that $\mathbb{A}u \in \mathrm{L}^1(B, W)$ with the obvious norm. In view of characterising operators $\mathbb{A}$ admitting inequalities of the form (1.8), we confine to operators of order one first and state our main result in the following slightly more elaborate form:

Theorem 1.1. *Let $\mathbb{A}$ be as in (1.4), $k = 1$, $n > 1$. The following are equivalent:*

(a) $\mathbb{A}$ *is $\mathbb{C}$–elliptic.*
(b) $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(B, V)$.
(c) $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^p(B, V)$ *for some* $1 < p \leqslant \frac{n}{n-1}$.
(d) $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^q(B, V)$ *for all* $1 \leqslant q < \frac{n}{n-1}$.
(e) $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^1(B, V)$.

This result manifests the following dichotomy: Either $\mathbb{A}$ is $\mathbb{C}$-elliptic, in which case one retrieves the known Sobolev embeddings on domains for the gradient or symmetric gradients, say, or $\mathbb{A}$ is *not* $\mathbb{C}$-elliptic. In this case estimates trivialise in the sense that for such $\mathbb{A}$, $u \in \mathrm{W}^{\mathbb{A},1}(B)$ in general only belongs to $\mathrm{L}^1(B; V)$ (which holds by definition of $\mathrm{W}^{\mathbb{A},1}$) but *no better* L^p*-space*.

To prove this, a crucial step is to show that $\mathbb{C}$–ellipticity implies cancellation. Then we extend to full–space and employ (1.3). The fact that the sub–critical embedding in (d) is compact generalizes the well–known result for BD (i.e., for $\mathbb{A} = \mathcal{E}$; see [23, 43, 45]) and is achieved by a careful application of the Riesz–Kolmogorov criterion. It is a priori far from obvious how can one connect the algebraic definitions of $\mathbb{C}$–ellipticity (1.6) and cancellation (1.5). We, however, consider the analytic characterization of cancellation [52, Prop. 6.1] and a consequence of Smith's representation formulas [40] (the kernel of a $\mathbb{C}$–elliptic operator is finite dimensional) to build a bridge in Lemma 3.2. In fact, it was already observed in [12] for first order operators that $\mathbb{C}$–ellipticity of $\mathbb{A}$ is equivalent with

$$\dim\{u \in \mathscr{D}'(\mathbb{R}^n, V) \colon \mathbb{A}u = 0\} < \infty. \tag{FDN}$$

In Proposition 3.1 we will give a short proof of the fact that $\mathbb{C}$–ellipticity and FDN are equivalent for operators of arbitrary order. Henceforth, we will thus use "FDN" and "$\mathbb{C}$–elliptic" interchangeably. On the other hand, we also show that the implication of EC by $\mathbb{C}$–ellipticity is strict by considering the first order operator

$$\mathbb{A}u = (\partial_1 u_1 - \partial_2 u_2,\, \partial_2 u_1 + \partial_1 u_2,\, \partial_3 u_1,\, \partial_3 u_2) \qquad \text{for } u \colon \mathbb{R}^3 \to \mathbb{R}^2.$$

	$N=1$	$N\geq 2$
$n=2$	$k=1$: E$\Rightarrow$FDN $k=2$: EC$\Rightarrow$FDN $k\geq 3$: EC$\not\Rightarrow$FDN	$k=1$: EC$\Rightarrow$FDN $k\geq 2$: EC$\not\Rightarrow$FDN
$n\geq 3$	$k=1$: E$\Rightarrow$FDN $k\geq 2$: EC$\not\Rightarrow$FDN	EC$\not\Rightarrow$FDN

FIGURE 1. Relationships between EC (elliptic and cancelling) and FDN (finite dimensional nullspace) for all constellations of $n\geq 2$, $N\geq 1$ and the order $k\geq 1$ of $\mathbb{A}$.

In particular, for this operator, there are maps in $\mathrm{W}^{\mathbb{A},1}(B)$ that are locally $\frac{n}{n-1}$–integrable, but are *not* L^p–integrable up to the boundary for any $p>1$. We further expand on these points in Section 3.1.

In Section 3 we will give a more comprehensive comparison of the two conditions, depending on n, $N=\dim V$ and the order k of $\mathbb{A}$; this is depicted in Figure 1. In general, we have the following result, which we believe to be of independent interest besides serving as a crucial tool in the proof of Theorem 1.1:

Theorem 1.2. *Let $\mathbb{A}$ be as in (1.4) with $k\geq 1$ and $n>1$. Then $\mathbb{A}$ has FDN if and only if $\mathbb{A}$ is $\mathbb{C}$–elliptic. Moreover, if $\mathbb{A}$ has FDN, then $\mathbb{A}$ is elliptic and cancelling, and there exists a bounded, linear extension operator $E_B\colon \mathrm{W}^{\mathbb{A},1}(B)\to\mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$.*

Due to lack of boundedness of singular integrals on L^1, the matter of extending $\mathrm{W}^{\mathbb{A},1}(B)$-maps is much more delicate as in the case of usual Sobolev spaces (or in the more general contexts considered in [40, 30]), cp. Lemma 5.7. Instead, we resort to the technique introduced by JONES [28]. From a conceptual perspective, this method crucially relies on inverse estimates for polynomials and thereby underlines the need of the FDN. Using the tools from Theorem 1.2, we can refine our result on fractional scales, thereby obtaining the local versions of the embeddings in [52, Thm. 8.1, Thm. 8.4]:

Theorem 1.3. *Let $\mathbb{A}$ be as in (1.4) with $k\geq 1$, $s\in[k-1,k)$ and $q\in(1,\infty)$. Then $\mathbb{A}$ has FDN if and only if there exists $c>0$ such that*

$$\|u\|_{\mathrm{B}_q^{s,\frac{n}{n-k+s}}(B,V)}\leqslant c\left(\|\mathbb{A}u\|_{\mathrm{L}^1(B,W)}+\|u\|_{\mathrm{L}^1(B,V)}\right)$$

for all $u\in\mathrm{C}^\infty(\bar{B},V)$.

Here, the Besov spaces on domains are defined as in [17, Sec. 2]. We obtain the embeddings $\mathrm{W}^{\mathbb{A},1}(B)\hookrightarrow\mathrm{W}^{s,n/(n-k+s)}(B,V)$ if we choose $q=n/(n-k+s)$ (cp. [52, Thm. 8.1]) and $\mathrm{W}^{\mathbb{A},1}(B)\hookrightarrow\mathrm{W}^{k-1,n/(n-1)}(B,V)$ if we further choose $s=k-1$ (cp. [52, Thm. 1.3]). The novelty of Theorems 1.1 and 1.3 comes from the fact that, up to our knowledge, there are only a few examples of L^1–estimates near the boundary that go in the direction of [53, Open Prob. 3] (cp. [15]). The result of Theorem 1.3 is sharp on the fractional (or Besov) scale, in the sense that the parameter $s=k$ is ruled out by Ornstein's Non–Inequality. Other embeddings into scales as Lorentz spaces are possible, too, but can be obtained in the same way as the aforementioned embedding into Besov spaces, now using Theorem 1.2 in conjunction with the results of [52].

1.3. **Organisation of the paper.** This paper is organized as follows: In Section 2 we collect preliminaries on function spaces, multi–linear algebra, harmonic analysis and give examples of operators. In Section 3 we give the proof of the first two statements in Theorem 1.2 and complete the comparison between EC and FDN, as well as the comparison between the embeddings (1.3) and (1.8). In Section 4 we construct the Jones–type extension and prove Theorems 1.1 and 1.3.

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2. Preliminaries

Throughout this paper we assume that $n>1$.

2.1. **Function spaces.** We define, reminiscent of [33], for $1\leqslant p\leqslant\infty$ and open $\Omega\subset\mathbb{R}^n$

$$\begin{aligned}\mathrm{W}^{\mathbb{A},p}(\Omega)&:=\{u\in\mathrm{L}^p(\Omega,V)\colon \mathbb{A}u\in\mathrm{L}^p(\Omega)\},\\ \mathrm{BV}^{\mathbb{A}}(\Omega)&:=\{u\in\mathrm{L}^1(\Omega,V)\colon \mathbb{A}u\in\mathcal{M}(\Omega,W)\},\\ \mathrm{V}^{\mathbb{A},p}(\Omega)&:=\{u\in\mathrm{W}^{\mathbb{A},p}(\Omega)\colon \nabla^l u\in\mathrm{L}^p(\Omega,V\odot^l\mathbb{R}^n),l=1\dots k-1\},\end{aligned}$$

and the homogeneous spaces $\dot{\mathrm{W}}^{\mathbb{A},p}$ as the closure of $\mathrm{C}_c^\infty(\mathbb{R}^n,V)$ in the semi–norm $|u|_{\mathbb{A},p}:=\|\mathbb{A}u\|_{\mathrm{L}^p}$. In the case $\mathbb{A}=\nabla^k$, we write $\mathrm{W}^{k,p}(\Omega,V)$, $\mathrm{V}^{k,p}(\Omega,V)$. When it is clear from the context what the target space is, we abbreviate the L^p–norm of maps defined on Ω by $\|\cdot\|_{p,\Omega}$. We denote the space of V–valued polynomials of degree at most d in n variables by $\mathbb{R}_d[x]^V$. We recall the weighted Bergman spaces $A^p_\alpha(\mathbb{D})$ of holomorphic maps defined on the open unit disc $\mathbb{D}\subset\mathbb{C}$, that are p–integrable with weight $w_\alpha(z)=(1-|z|^2)^\alpha$. It is well–known that these are Banach spaces under the $\mathrm{L}^p_{w_\alpha}$–norm for $1\leqslant p<\infty$ and $-1<\alpha<\infty$. We also recall, for $s>0$, $1\leqslant p,q<\infty$, the Besov space

$$\mathrm{B}^{s,p}_q(\Omega):=\{u\in\mathrm{L}^p(\Omega)\colon |u|_{\mathrm{B}^{s,p}_q(\Omega)}<\infty\},$$

with an obvious choice of norm. Here, the Besov semi–norm is defined (see, e.g., [17, Sec. 2]) for integer $r>s$ by

$$|u|_{\mathrm{B}^{s,p}_q(\Omega)}=\|u\|_{\dot{\mathrm{B}}^{s,p}{}_q(\Omega)}:=\left(\int_0^\infty\frac{\sup_{|h|<t}\|\Delta^r_h u\|^q_{\mathrm{L}^p(\Omega)}}{t^{1+sq}}\,\mathrm{d}t\right)^{\frac{1}{q}},$$

where the r-th finite difference $\Delta^r_h u$ is defined to be zero if undefined, i.e., if at least one of $x+jh$, $j=1\dots r$, falls outside Ω. We also define the homogeneous space $\dot{\mathrm{B}}^{s,p}{}_q(\mathbb{R}^n)$ as the closure of $\mathrm{C}_c^\infty(\mathbb{R}^n)$ in the Besov semi–norm.

We also collect the assumptions on our operators. As in Section 1, we say that $\mathbb{A}$ is $(\mathbb{C}-)$*elliptic* if and only if the linear map $\mathbb{A}[\xi]\colon V(+\mathrm{i}\,V)\to W(+\mathrm{i}\,W)$ is injective for all non–zero $\xi\in\mathbb{R}^n(+\mathrm{i}\,\mathbb{R}^n)$. Trivially, $\mathbb{C}$–elliptic operators are elliptic. We say that $\mathbb{A}$ has *FDN* (finite dimensional null–space) if and only if the vector space $\{u\in\mathscr{D}'(\mathbb{R}^n,V)\colon\mathbb{A}u=0\}$ is finite dimensional. Finally, $\mathbb{A}$ is *cancelling* if and only if $\bigcap_{\xi\in S^{n-1}}\mathbb{A}[\xi](V)=\{0\}$.

2.2. **Multi-linear algebra.** Let U,V be finite dimensional vector spaces and $l\in\mathbb{N}$. We write $\mathscr{L}(U,V)$ for the space of linear maps $U\to V$ and $V\odot^l U$ for the space of V–valued symmetric l–linear maps on U. This is naturally the space of the l–th gradients, i.e., $D^l f(x)\in V\odot^l U$ for $f\in\mathrm{C}^l(U,V)$, $x\in U$. For more detail, see [20, Ch. 1]. We also write $a\otimes b=(a_ib_j)$ (the usual tensor product) and $\otimes^l a:=\otimes a\otimes\ldots\otimes a$, where a appears l times on the right hand side. We single out the standard fact that $\widehat{\nabla^l f}(\xi)=\hat{f}(\xi)\otimes^l\xi\in V\odot^l U$ for $f\in\mathscr{S}(U,V)$, $\xi\in U$. We recall the pairing introduced in [12], $v\otimes_{\mathbb{A}}\xi:=\mathbb{A}[\xi]v$, which is reminiscent of the tensor product notation, i.e., if $\mathbb{A}=D$, we have $\otimes_{\mathbb{A}}=\otimes$. We have the following for $k=1$:

$$\begin{aligned}\mathbb{A}(\rho u)&=\rho\mathbb{A}u+u\otimes_{\mathbb{A}}\nabla\rho &&\text{for } u\in\mathrm{C}^1(\mathbb{R}^n,V),\ \rho\in\mathrm{C}^1(\mathbb{R}^n),\\ \mathbb{A}(\phi(w))&=\phi'(w)\otimes_{\mathbb{A}}\nabla w &&\text{for } \phi\in\mathrm{C}^1(\mathbb{R},V),\ w\in\mathrm{C}^1(\mathbb{R}^n).\end{aligned}$$

The above can easily be checked by direct computation and will be used without mention.

2.3. **Harmonic analysis.** Let $\mathbb{A}$ as in (1.4) be elliptic and $u \in \mathscr{S}(\mathbb{R}^n, V)$. We Fourier transform $\mathbb{A}u$ and apply the one–sided inverse $m_{\mathbb{A}}(\xi) := (\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi] \in \mathscr{L}(W, V)$ of $\mathbb{A}[\xi]$ to get that $\hat{u}(x) = m_{\mathbb{A}}(\xi)\widehat{\mathbb{A}u}(\xi)$ for $\xi \in \mathbb{R}^n$ (we omitted the complex multiplicative constant arising from Fourier transforming, as it can be absorbed in the definition of $m_{\mathbb{A}}$). We define the map $\mathbf{G}_{\mathbb{A}}$ as the inverse Fourier transform of the k–homogeneous map $m_{\mathbb{A}}$. Thus we have the Green's function representation $u = \mathbf{G}_{\mathbb{A}} \star \mathbb{A}u$. These considerations are formalized in [11, Lem. 2.1], an implication of which we recall below:

Lemma 2.1. *Let $\mathbb{A}$ as in (1.4) be elliptic. Then there exists a $(1-n)$–homogeneous map $\mathbf{K}_{\mathbb{A}} \in \mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}, \mathscr{L}(W, V \odot^{k-1} \mathbb{R}^n))$ such that*

$$D^{k-1}u(x) = \int_{\mathbb{R}^n} \mathbf{K}_{\mathbb{A}}(x-y)\mathbb{A}u(y)\,\mathrm{d}y = (\mathbf{K}_{\mathbb{A}} \star \mathbb{A}u)(x) \tag{2.1}$$

for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$.

We also record standard facts regarding L^p–boundedness of Riesz potentials (see [42, Ch. V.1] and [25, Lem. 7.2]), which are defined by

$$I_\alpha f := |\cdot|^{\alpha-n} \star f$$

for $\alpha \in [0, n)$ and measurable $f\colon \mathbb{R}^n \to \mathbb{R}$.

Theorem 2.2. *Let $1 \leqslant p, q \leqslant \infty$. We have that:*

(a) *I_α is bounded $\mathrm{L}^p(\mathbb{R}^n) \to \mathrm{L}^q(\mathbb{R}^n)$ for $0 < \alpha < n$, $1 < p < n/\alpha$, $q = np/(n-\alpha p)$;*
(b) *I_α is bounded $\mathrm{L}^p(\Omega) \to \mathrm{L}^q(\Omega)$ for $0 < \alpha < n$, $0 \leqslant n(1/p - 1/q) < \alpha$ with*

$$\|I_\alpha f\|_{\mathrm{L}^q(\Omega)} \leqslant c(\operatorname{diam}\Omega)^{\alpha - n(1/p-1/q)}\|f\|_{\mathrm{L}^p(\Omega)}$$

for all $f \in \mathrm{L}^p(\Omega)$.

In Theorem 2.2(b) we make the convention $1/\infty = 0$.

2.4. **Examples.** We give examples of operators arising in conductivity, elasticity, plasticity and fluid mechanics ([21, 23, 34]). Let $\mathbb{A}$ be as in (1.4). The facts that we use without mention are the main Theorems 1.1, 1.2, and 1.3.

(a) If $\mathbb{A} = \nabla^k$, we have that $\ker \mathbb{A} = \mathbb{R}_{k-1}[x]^V$, so $\mathbb{A}$ has FDN, hence is EC. This, of course, corresponds to the case of classical Sobolev spaces, but we highlight it here to stress that our generalization brings a new perspective on their study.
(b) If $\mathbb{A}u = \mathcal{E}u := (\nabla u + (\nabla u)^\mathsf{T})/2$ is the symmetrized gradient, it is easy to see that $\ker \mathbb{A}$ is the space of rigid motions, i.e., affine maps of anti–symmetric gradient, so $\mathbb{A}$ has FDN, hence is EC. In this case, we recover the inequality in [43, Prop. 1.2].
(c) Let $\mathbb{A}u = \mathcal{E}^D u := \mathcal{E}u - (\operatorname{div} u/n)\mathbf{I}$, where $n \geq 2$ and $\mathbf{I}$ is the identity $n \times n$ matrix. If $n \geq 3$, we have from [38] that $\ker \mathbb{A}$ is the space of conformal Killing vectors, so $\mathbb{A}$ has FDN, hence is EC. If $n = 2$, we show in Counterexample 3.4 that $\mathbb{A}$ is elliptic. However, under the canonical identification $\mathbb{R}^2 \cong \mathbb{C}$, we can also identify $\mathcal{E}^D$ with the anti–holomorphic derivative $\bar{\partial}$, so that we can further identify $\ker \mathbb{A}$ with the space of holomorphic functions, so $\mathbb{A}$ does not have FDN. Neither is $\mathbb{A}$ cancelling: by ellipticity, we have that $\mathcal{E}^D[\xi](\mathbb{R}^2) = \mathbb{R}^2$. No critical embedding (3.1), (3.2) can hold in this case.
(d) If $\mathbb{A} = \Delta$, which is clearly elliptic, we have that $\ker \mathbb{A}$ is the space of all harmonic functions, so $\mathbb{A}$ does not have FDN and since $\mathbb{A}[\xi](V) = (\xi_1^2 + \ldots + \xi_n^2)\mathbb{R}^N = \mathbb{R}^N$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, neither is $\mathbb{A}$ cancelling.
(e) If $\mathbb{A}$ is elliptic, one can consider minimizers of the $\mathbb{A}$–Dirichlet energy $u \mapsto \int_B |\mathbb{A}u|^2\,\mathrm{d}x$, which has Euler–Lagrange system $\mathbb{A}^*\mathbb{A}u = 0$. Then $\Delta_{\mathbb{A}} := \mathbb{A}^*\mathbb{A}$ is elliptic, as $\langle(\mathbb{A}^*\mathbb{A})[\xi]v, v\rangle = |\mathbb{A}[\xi]v|^2 \gtrsim |\xi|^{2k}|v|^2$, where the last inequality follows from $|\mathbb{A}[\xi]v| > 0$ on $\{|\xi| = 1, |v| = 1\}$ and homogeneity. Therefore $(\mathbb{A}^*\mathbb{A})[\xi](V) = V$ for all $\xi \neq 0$, so the Euler–Lagrange system above has infinite dimensional solution space (by Lemma 3.2).

3. EC Versus FDN

We begin by proving the first two statements in Theorem 1.2. Throughout, $n > 1$.

Proposition 3.1. *Let $\mathbb{A}$ be as in (1.4). Then $\mathbb{A}$ has FDN if and only if $\mathbb{A}$ is $\mathbb{C}$–elliptic.*

Proof. From Theorem 5.3, we have that if $\mathbb{A}$ is $\mathbb{C}$–elliptic, then $\ker\mathbb{A}$ consists of polynomials of fixed maximal degree. Suppose now that $\mathbb{A}$ is not $\mathbb{C}$–elliptic, so that there exist non–zero $\xi \in \mathbb{C}^n$, $v \in V + \mathrm{i}\,V$ such that $\mathbb{A}[\xi]v = 0$. We define $u_f(x) = f(x\cdot\xi)v$, for holomorphic $f\colon \mathbb{C} \to \mathbb{C}$. It can be shown by direct real differentiation of real and imaginary parts and use of the Cauchy–Riemann equations for f that $Du_f(x) = (\partial_1 f)(x\cdot\xi)v \otimes \xi$. Since $\partial_1 f$ is itself holomorphic, inductively we get that $D^l u_f(x) = (\partial_1^l f)(x\cdot\xi)v \otimes^l \xi$. We make the simple observation that there exists a linear map $A \in \mathscr{L}(V \odot^k \mathbb{R}^n, W)$ such that $\mathbb{A}u = A(D^k u)$, which can be viewed as a coordinate invariant (jet) definition of $\mathbb{A}$. In this notation, by standard properties of the Fourier transform we get $\mathbb{A}[\eta]w = A(w \otimes^k \eta)$ for $\eta \in \mathbb{R}^n$, $w \in V$. It is then easy to see that $\mathbb{A}u_f(x) = (\partial_1^k f)(x\cdot\xi)A(v \otimes^k \xi) = 0$. In particular, $\Re u_f, \Im u_f \in \ker\mathbb{A}$, so $\mathbb{A}$ has infinite dimensional null–space. $\square$

The above result enables us to use FDN and $\mathbb{C}$–ellipticity interchangeably. Note that to prove that FDN implies ellipticity, one can simply take real ξ, v and $f \in \mathrm{C}^1(\mathbb{R})$. We next provide an instrumental ingredient for proving sufficiency of FDN for Theorem 1.3.

Lemma 3.2. *Let $\mathbb{A}$ be as in (1.4). If $\mathbb{A}$ has FDN, then $\mathbb{A}$ is cancelling.*

Proof. We use Lemma 5.1. Let $u \in \mathrm{C}^\infty(\mathbb{R}^n, V)$ be such that $K := \operatorname{spt}\mathbb{A}u$ is compact. Consider an open ball B containing K. Cover the complement of B with an increasing chain of overlapping open balls B_j such that $B^c \subset \bigcup_j B_j \subset K^c$. In particular, we have $\mathbb{A}u = 0$ in each B_j, so by Theorem 5.3, u must be a polynomial of degree at most $d(\mathbb{A})$ in each B_j. Since the pairs of balls overlap on a set of positive measure, we get that u equals a V–valued polynomial P (tacitly viewed as already extended to the entire $\mathbb{R}^n$) in B^c such that $\mathbb{A}P = 0$ in $\mathbb{R}^n$. To conclude, we elaborate on the notation introduced in the proof of Proposition 3.1. Put $m := \dim W$, so that we can write in coordinates $(A\mathscr{V})_l = A^l \cdot \mathscr{V}$ for fixed $A^l \in V \odot^k \mathbb{R}^n$, $l = 1\dots m$, and all $\mathscr{V} \in V \odot^k \mathbb{R}^n$. For $l = 1\dots m$, we integrate by parts to get

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathbb{A}u)_l \,\mathrm{d}x &= \int_B A^l \cdot D^k u \,\mathrm{d}x = \int_{\partial B} A^l \cdot (D^{k-1}u \otimes \nu)\,\mathrm{d}\mathcal{H}^{n-1}\\ &= \int_{\partial B} A^l \cdot (D^{k-1}P \otimes \nu)\,\mathrm{d}\mathcal{H}^{n-1} = \int_B A^l \cdot D^k P \,\mathrm{d}x = \int_B (\mathbb{A}P)_l \,\mathrm{d}x = 0,\end{aligned}$$

where ν denotes the unit normal to ∂B. The proof is complete. $\square$

The converse of Lemma 3.2, however, is not true in general. In what follows, we complete the algebraic comparison of the FDN condition and Van Schaftingen's EC condition. We write $N := \dim V$. The streamline here is that for $N = k = 1$, ellipticity alone implies FDN (rendering these cases rather uninteresting), whereas in higher dimensions or for higher orders, there are EC operators that are not FDN. Somewhat surprisingly, there are also a few instances in which ellipticity and $\mathbb{C}$–ellipticity differ, but EC implies FDN. We give the details below.

Lemma 3.3. *Let $\mathbb{A}$ as in (1.4) be elliptic, $N = k = 1$. Then $\mathbb{A}$ has FDN.*

Proof. Since $N = 1$, it is clear that $\mathbb{A}$ is $\mathbb{F}$–elliptic, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, if and only if the polynomials $(\mathbb{A}[\xi])_l$, $l = 1\dots m$, have no common non–trivial zeroes in $\mathbb{F}$. Since we also assume $k = 1$, we have $\mathbb{A}[\xi] = A\xi$ for some $A \in \mathbb{R}^{m\times n}$ It is clear that all roots of the polynomials thus arising are real (in fact, $\mathbb{A}$ is $\mathbb{F}$–elliptic if and only if $\ker_{\mathbb{R}} A = 0$). $\square$

If $n \geq 3$, EC turns out to be insufficient for FDN, even for scalar fields or first order operators.

Counterexample 3.4 (EC does *not* imply FDN)**.** *Consider the operators*

$$\mathbb{A}_{k,n}u := \nabla^{k-1}\left(\partial_1 u_1 - \partial_2 u_2, \partial_2 u_1 + \partial_1 u_2, \partial_j u_i\right)_{(i,j)\notin\{1,2\}\times\{1,2\}} \qquad \text{for } N \geq 2,$$
$$\mathbb{B}_{k,n}u := \nabla^{k-2}\left(\partial_1^2 u + \partial_2^2 u, \partial_j^2 u\right)_{j=3\ldots n} \qquad \text{for } N = 1, k \geq 2.$$

If $n \geq 3$ or $k \geq 2$, then $\mathbb{A}_{k,n}$ is elliptic and cancelling, but has infinite dimensional null–space. The same is true of $\mathbb{B}_{k,n}$ if $n \geq 3$ or $k \geq 3$.

Proof. The failure of FDN is clear: simply take

$$u_{\mathbb{A}}(x) := \left(\Re f\left(x_1 + \mathrm{i}\, x_2\right), \Im f\left(x_1 + \mathrm{i}\, x_2\right), 0, \ldots, 0\right)^{\mathsf{T}}$$
$$u_{\mathbb{B}}(x) := g(x_1, x_2)$$

for holomorphic f and (scalar) harmonic g. We next show that $\mathbb{A}_{k,n} = \nabla^{k-1}\mathbb{A}_{1,n}$ is elliptic if $n, N \geq 2$. We can reduce to ellipticity of $\mathbb{A}_{1,n}$, since for non–zero ξ, we have that $0 = \mathbb{A}_{k,n}[\xi]v = (\mathbb{A}_{1,n}[\xi]v)\otimes^{k-1}\xi$, so $\mathbb{A}_{1,n}[\xi]v = 0$. Let $1 \leqslant j \leqslant n$ be such that $\xi_j \neq 0$. If $j \geq 3$, we clearly get $v = 0$. If $1 \leqslant j \leqslant 2$, we get that $v_i = 0$ for $3 \leqslant i \leqslant N$. The remaining equations are $\xi_1 v_1 - \xi_2 v_2 = 0 = \xi_2 v_1 + \xi_1 v_2$, with determinant $\xi_1^2 + \xi_2^2 > 0$, so $v_1 = 0 = v_2$. It remains to check that, under our assumptions, $\mathbb{A}_{k,n}$ is cancelling. The case $k > 1$ is easier, since the composition of operators $\mathbb{L}_1 \circ \mathbb{L}_2$ is cancelling if $\mathbb{L}_1$ is. This is simply due to the fact that $\mathrm{im}(\mathbb{L}_1 \circ \mathbb{L}_2)[\xi] = \mathbb{L}_1[\xi](\mathrm{im}\mathbb{L}_2[\xi]) \subseteq \mathrm{im}\mathbb{L}_1[\xi]$. If $k = 1$ and $n \geq 3$ we can make a straightforward computation. Write $(w_l)_{l=1\ldots Nn-2} := \mathbb{A}_{1,n}[\xi]v$. For $w \in \bigcap_{\xi\neq 0}\mathbb{A}_{1,n}[\xi](V)$, we can essentially test with different values of $\xi \neq 0$. By choosing ξ to have exactly one non–zero entry, we obtain that $w_l = 0$ for $3 \leqslant l \leqslant Nn-2$. Incidentally, when testing with ξ such that $\xi_1 = 0 = \xi_2$, we also obtain $w_1 = 0 = w_2$, so all properties are checked for $\mathbb{A}_{k,n}$. Ellipticity of $\mathbb{B}_{k,n}$ is obvious, whereas cancellation is established analogously. □

The two specific cases that are not covered by Lemma 3.3 and Counterexample 3.4 reveal that the classes EC and FDN can coincide even if they are strictly smaller than the class of elliptic operators.

Lemma 3.5. *Let $n = 2$ and $\mathbb{A}$ be as in (1.4) be elliptic but not $\mathbb{C}$–elliptic. If any of the following hold,*

(a) $N = 1$, $k = 2$,
(b) $N \geq 2$, $k = 1$,

then $\mathbb{A}$ is not cancelling.

Proof. Suppose that (a) holds. Since $N = 1$ and $\mathbb{A}$ is not $\mathbb{C}$–elliptic, the homogeneous, quadratic, scalar polynomials $(\mathbb{A}[\xi])_l$, $l = 1\ldots m$, must have a common complex root. This root cannot be real, as $\mathbb{A}$ is real–elliptic. It follows that $(\mathbb{A}[\xi])_l$ are all multiples of the same quadratic polynomial $P : \mathbb{R} \to \mathbb{R}$, so that $\mathbb{A}[\xi]v = vP(\xi)w_0$ for all $v \in V \simeq \mathbb{R}$ and some $w_0 \in W \setminus \{0\}$. It is clear then that $\mathbb{A}[\xi](V) = \mathbb{R}w$ for all $\xi \neq 0$. We next assume that (b) holds. Since $\mathbb{A}$ is elliptic, there exist linearly independent $\xi, \eta \in \mathbb{R}^2$, $v, w \in \mathbb{R}^N$ such that $\mathbb{A}[\xi]v = \mathbb{A}[\eta]w$ and $\mathbb{A}[\xi]w = -\mathbb{A}[\eta]v$. We also have that any $\zeta \in \mathbb{R}^2$ can be written as $\zeta = a\xi + b\eta$. We put $v_\zeta := av + bw$. It follows that

$$\mathbb{A}[\zeta]v_\zeta = \mathbb{A}[a\xi + b\eta](av + bw) = (a^2 + b^2)\mathbb{A}[\xi]v,$$

so that $\bigcap_{\zeta\in\mathbb{R}^2\setminus\{0\}}\mathbb{A}[\zeta](V) \ni \mathbb{A}[\xi]v \neq 0$. □

We conclude this section with a minor curiosity: we can append the proof above by taking $w_\zeta := bv - aw$ and obtain $\bigcap_{\zeta\neq 0}\mathbb{A}[\zeta](V) \supset \{\mathbb{A}[\xi]v, \mathbb{A}[\xi]w\}$. Therefore, if $n = 2$, $k = 1$, and $\mathbb{A}$ is elliptic but not cancelling, then

$$\dim \bigcap_{\zeta\neq 0}\mathbb{A}[\zeta](V) \geq 2.$$

This is no longer the case in higher dimensions, as can be seen by considering the first order operator $\mathbb{A}u = (\operatorname{div} u, \operatorname{curl} u)$ for $u\colon \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 3$.

3.1. **Insufficiency of EC.** We next give examples of first order EC operators and domains $\Omega \subset \mathbb{R}^n$ for which the Sobolev–type embedding fails. Firstly, we pause to compare the embeddings in Theorem 1.1(b), (c) with VAN SCHAFTINGEN's homogeneous embedding $\dot{\mathrm{W}}^{\mathbb{A},1}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n, V)$. For elliptic $\mathbb{A}$, the latter embedding is equivalent to

$$\mathrm{W}^{\mathbb{A},1}_0(B) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(B, V) \tag{3.1}$$

(see Lemma 5.9 for a scaling argument) and it can easily be shown that, in the absence of cancellation, we can still prove by means of a Green's formula and boundedness of Riesz potentials that $\mathrm{W}^{\mathbb{A},1}_0(B) \hookrightarrow \mathrm{L}^p(B, V)$ for any $1 \leqslant p < n/(n-1)$ (see Lemma 5.10). Here $\mathrm{W}^{\mathbb{A},p}_0(B)$ is defined as the closure of $\mathrm{C}^\infty_c(B, V)$ in the (semi–)norm $u \mapsto \|\mathbb{A}u\|_{\mathrm{L}^p}$. The situation is dramatically different as far as L^p–embeddings of $\mathrm{W}^{\mathbb{A},1}(B)$ are concerned. By Theorem 1.1, if the critical embedding

$$\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(B, V) \tag{3.2}$$

fails, then no uniform higher integrability estimate is possible. The difference can be even sharper: for EC, non–FDN operators there are maps in $\mathrm{W}^{\mathbb{A},1}(B)$ that have no higher integrability, so the homogeneous embedding (3.1) can hold even if the inhomogeneous (3.2) fails completely. We highlight that the main difference between $\mathrm{W}^{\mathbb{A},1}(B)$ and $\mathrm{W}^{\mathbb{A},1}_0(B)$ lies in the traces, which are integrable if and only if $\mathbb{A}$ has FDN [12].

The existence of elliptic and cancelling $\mathbb{A}$, domains Ω, and of maps $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$ that are in no $\mathrm{L}^p(\Omega, V)$, $p > 1$, follows from Counterexample 3.4 above and the next Lemma, which is a strengthened version of the strict inclusion of (weighted) Bergman spaces generalized to elliptic, non–FDN operators.

Lemma 3.6. *Let $k = 1$ and $\mathbb{A}$ as in (1.4) be elliptic but* not *have FDN, so there exist linearly independent $\eta_1, \eta_2 \in \mathbb{R}^n$ such that $\mathbb{A}[\eta_1 + \mathrm{i}\,\eta_2]$ has non–trivial kernel in $V + \mathrm{i}\,V$. Assume that η_1, η_2 are orthonormal. If any of the following holds:*

(a) $\Omega := B_{\mathrm{span}\{\eta_1,\eta_2\}} \times [0, 1]^{n-2}$,
(b) $\Omega := B$,

then there exists smooth $u \in \mathrm{L}^1 \setminus \bigcup_{p>1} \mathrm{L}^p(\Omega, V)$ such that $\mathbb{A}u = 0$.

Proof. We write $\xi = \eta_1 + \mathrm{i}\,\eta_2$, and write D for the unit disc in $\mathrm{span}\{\eta_1, \eta_2\}$. We stress that each η_j must be non–zero by ellipticity of $\mathbb{A}$, so D is indeed a non–degenerate disc. We also know from the proof of Proposition 3.1 that there exist non–zero $v \in V + \mathrm{i}\,V$ such that $\mathbb{A}[\xi]v = 0$, and one can show by direct computation that for any holomorphic function f we can define $u_f(x) := f(x \cdot \xi)v$, for which $\mathbb{A}\Re u_f = 0 = \mathbb{A}\Im u_f$. We have that

$$\begin{aligned}\int_\Omega |u_f(x)|^p \,\mathrm{d}x &= \int_D \int_{(\eta+\{\eta_1,\eta_2\}^\perp)\cap\Omega} |f(\eta\cdot\xi)|^p |v|^p \,\mathrm{d}\mathcal{H}^{n-2}\,\mathrm{d}\mathcal{H}^2(\eta)\\ &= |v|^p \int_D |f(\eta\cdot\xi)|^p \mathcal{H}^{n-2}\left((\eta + \{\eta_1,\eta_2\}^\perp)\cap\Omega\right)\mathrm{d}\mathcal{H}^2(\eta)\end{aligned}$$

We now make the case distinction. Assume (a) holds, so

$$\int_\Omega |u_f(x)|^p \,\mathrm{d}x = |v|^p \int_D |f(\eta\cdot\xi)|^p \,\mathrm{d}\mathcal{H}^2(\eta) = \int_{\mathbb{D}} |f(z)|^p \,\mathrm{d}\mathcal{L}^2(z).$$

Assume (b), so

$$\begin{aligned}\int_\Omega |u_f(x)|^p \,\mathrm{d}x &= c(n)|v|^p \int_D |f(\eta\cdot\xi)|^p (1-|\eta|^2)^{\frac{n-2}{2}} \,\mathrm{d}\mathcal{H}^2(\eta)\\ &= c(n)|v|^p \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{\frac{n-2}{2}} \,\mathrm{d}\mathcal{L}^2(z),\end{aligned}$$

where $c(n)$ denotes the volume of the $(n-2)$–dimensional ball. By Lemma 3.7 below, we can choose $f \in A^1_\alpha(\mathbb{D}) \setminus \bigcup_{p>1} A^p_\alpha(\mathbb{D})$ for $\alpha = 0$ and $\alpha = (n-2)/2$ respectively, so that both $\Re u_f$ and $\Im u_f$ are in $\mathrm{L}^1(B, V)$, but one of them is in not in any other L^p. This proves the claim. $\square$

The following Lemma is also feasible by direct computation, but we prefer to give an abstract argument for the sake of brevity.

Lemma 3.7. *For all $1 \leqslant p < \infty$, $\alpha \geq 0$ the set $A^p_\alpha(\mathbb{D}) \setminus \bigcup_{q>p} A^q_\alpha(\mathbb{D})$ is non-empty.*

Proof. We abbreviate $A^p := A^p_\alpha(\mathbb{D})$. The proof relies on the strict inclusion $A^q \subsetneq A^p$ for $1 \leqslant p < q < \infty$ proved in [55, Cor. 68] and a Baire category argument. Assume that the result is false, so that by Hölder's Inequality we can find a sequence $q_j \downarrow p$ such that $A^p = \bigcup_j A^{q_j}$. For natural l, we define the sets $F^j_l := \{f \in A^{q_j} : \|f\|^{q_j}_{A^{q_j}} \leqslant l\}$, which we claim to be closed in A^p. Let $f_m \in F^j_l$ converge to f in A^p. By completeness of A^p, we have, by Fatou's Lemma on a pointwise convergent, not relabelled subsequence that

$$\int_{\mathbb{D}} |f|^{q_j} w_\alpha \,\mathrm{d}\mathcal{L}^2 \leqslant \liminf_{m\to\infty} \int_{\mathbb{D}} |f_m|^{q_j} w_\alpha \,\mathrm{d}\mathcal{L}^2 \leqslant l,$$

so that indeed $f \in F^j_l$. Since A^{q_j} is a proper subspace of A^p, it follows that the sets F^j_l are nowhere dense in A^p. It remains to notice that then $A^p = \bigcup_{j,l} F^j_l$, which contradicts completeness of A^p by Baire's Theorem. □

3.2. **Comparison to the Bourgain–Brezis condition.** We recall here the assumptions on $\mathbb{A}$ (sufficient for EC) under which a general inequality of the type (1.3) was first proved in [7], in the case $k = 1$ and $V = \mathbb{R}^n$. In their notation, we write $(\mathbb{A}u)_s = \langle L^{(s)}, \nabla u\rangle$ for matrices $L^{(s)} \in \mathbb{R}^{n\times n}$, $s = 1\ldots m$. It is shown in [7, Thm. 25], that if an operator $\mathbb{A}$ is elliptic such that $\det L^{(s)} = 0$ for $s = 1\ldots m$, then (1.3) holds. It is clear (either by [52, Thm. 1.3] or by direct computation) that such operators are cancelling. By Lemma 3.5, if $n = 2$, we have that such $\mathbb{A}$ also has FDN, and thus satisfies (3.2). However, if $n \geq 3$, we show that $\mathbb{A}_{1,n}$ as in Counterexample 3.4 with $N = n$ satisfies the Bourgain–Brezis condition, but do not have FDN. We explicitly write down the matrices $L^{(s)}$ if $n = 3$, the general case being a simple exercise:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the reasoning in Section 3.1, with $\mathbb{A} = \mathbb{A}_{1,n}$, we have that $\dot{\mathrm{W}}^{\mathbb{A},1}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{n/(n-1)}(\mathbb{R}^n)$, but there are maps in $\mathrm{W}^{\mathbb{A},1}(B)$ that have no higher integrability.

4. The Sobolev–type Embedding on Domains

4.1. **A Jones–type Extension.** In this section we complete the proof of Theorem 1.2 with the following generalization:

Theorem 4.1. *Let $\mathbb{A}$ as in (1.4) have FDN, $1 \leqslant p < \infty$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a bounded, linear extension operator*

$$E_\Omega \colon \mathrm{W}^{\mathbb{A},p}(\Omega) \to \mathrm{V}^{\mathbb{A},p}(\mathbb{R}^n).$$

To prove this result we use Jones' method of extension developed in the celebrated paper [28]. Recall that Jones's original idea was to decompose a small neighbourhood of $\partial\Omega$ into small cubes and assign suitable polynomials of degree at most $k-1$ to each cube. Inspired by [12, Sec. 4.1-2], we assign elements of $\ker\mathbb{A}$ on such cubes, as explained below. We stress that the fact that $\ker\mathbb{A}$ consists of a finite dimensional space of polynomials is essential for the construction to work. With this crucial modification, the streamlined proof that we include below mostly follows the same lines as in [28, Sec. 2-3], where all the details we omit can be found. What deserves some special attention is a Poincaré–type inequality, which is interesting in its own right, as it implies that $\mathrm{W}^{\mathbb{A},p}(B) \simeq \mathrm{V}^{\mathbb{A},p}(B)$ for FDN operators (see Lemma 5.6). We present it below and mention that it is a generalization of the results in [33, Sec. 1.1.11] and [12, Sec. 3]. We extend the notation presented in Theorem 5.3 by $\pi_\Omega u := \Pi \mathcal{P} u$, where Π denotes the L^2–orthogonal projection of $\mathbb{R}_d[x]^V$ onto $\ker \mathbb{A}$.

Proposition 4.2 (Poincaré–type inequality). *Let $\mathbb{A}$ as in (1.4) have FDN, $1 \leqslant p \leqslant \infty$, $0 \leqslant l < k$, and $\Omega \subset \mathbb{R}^n$ be a star–shaped domain with respect to a ball. Then there exists $c > 0$ such that*

$$\|\nabla^l (u - \pi_\Omega u)\|_{p,\Omega} \leqslant c (\operatorname{diam} \Omega)^{k-l} \|\mathbb{A} u\|_{p.\Omega} \tag{4.1}$$

for all $u \in \mathrm{C}^\infty(\bar{\Omega}, V)$.

Interestingly, $\mathbb{A}$ having FDN is not necessary for the estimate (4.1) to hold, as can be seen from [22]. We believe that ellipticity alone is sufficient for the estimate to hold and intend to pursue this in future work.

Proof. We start with $\|\nabla^l (u - \pi_\Omega u)\|_{p,\Omega} \leqslant \|\nabla^l (u - \mathcal{P} u)\|_{p,\Omega} + \|\nabla^l (\mathcal{P} u - \pi_\Omega u)\|_{p,\Omega}$, and estimate both terms. We have by the growth conditions on K from Theorem 5.3 that

$$\begin{aligned} \|\nabla^l (u - \mathcal{P} u)\|_{p,\Omega} &= \left(\int_\Omega \left| \int_\Omega \nabla^l_x K(x,y) \mathbb{A} u(y) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} \\ &\lesssim \left(\int_\Omega \left(\int_\Omega \frac{|\mathbb{A} u(y)|}{|x-y|^{n+l-k}} \, \mathrm{d}y \right)^p \mathrm{d}x \right)^{\frac{1}{p}}. \end{aligned}$$

Now consider the case $n + l > k$, so that we can estimate the RHS using Theorem 2.2(b):

$$\|I_{k-l}(|\mathbb{A} u|)\|_{p,\Omega} \lesssim (\operatorname{diam} \Omega)^{k-l} \|\mathbb{A} u\|_{p,\Omega}.$$

If, on the other hand, $n + l \leqslant k$, with $R = \operatorname{diam} \Omega$, we perform the elementary estimation

$$\begin{aligned} \left(\int_\Omega \left(\int_\Omega |\mathbb{A} u(y)| |x-y|^{k-l-n} \, \mathrm{d}y \right)^p \mathrm{d}x \right)^{\frac{1}{p}} &\leqslant R^{k-l-n} \left(\int_\Omega |\mathbb{A} u(y)| \, \mathrm{d}y \right) \left(\int_\Omega \mathrm{d}x \right)^{\frac{1}{p}} \\ &\leqslant R^{k-l-n} \cdot R^{n(p-1)/p} \cdot \|\mathbb{A} u\|_{p,\Omega} \cdot R^{n/p} \\ &= R^{k-l} \|\mathbb{A} u\|_{p,\Omega}, \end{aligned}$$

where obvious modifications have to be made if $p = \infty$.

We then note that $P \mapsto \|P - \Pi P\|_{p,\Omega}$ and $P \mapsto \|\mathbb{A} P\|_{p,\Omega}$ respectively define a semi–norm and a norm on the finite dimensional vector space $\mathbb{R}_d[x]^V / \ker \mathbb{A}$, so that the second term $\|\nabla^l (\mathcal{P} u - \pi_\Omega u)\|_{p,\Omega} \lesssim \|\mathbb{A} \mathcal{P} u\|_{p,\Omega}$, with a domain dependent constant. We recall from the original proof of Theorem 5.3 that $\mathcal{P} u$ is the averaged Taylor polynomial

$$\mathcal{P} u(x) = \int_\Omega \sum_{|\alpha| \leqslant d} \frac{\partial_y^\alpha \left((y-x)^\alpha w(y) \right)}{\alpha!} u(y) \, \mathrm{d}y = \int_\Omega \sum_{|\alpha| \leqslant d} \frac{(x-y)^\alpha}{\alpha!} w(y) \partial^\alpha u(y) \, \mathrm{d}y,$$

where the weight w is a smooth map supported in the ball with respect to which Ω is star–shaped such that $\int w = 1$. One can show by direct computation that averaged Taylor polynomials "commute" with derivatives, in the sense that

$$\mathbb{A} \mathcal{P} u = \int_\Omega \sum_{|\beta| \leqslant d-k} \frac{\partial_y^\beta \left((y - \cdot)^\beta w(y) \right)}{\alpha!} \mathbb{A} u(y) \, \mathrm{d}y.$$

It is then obvious that $\|\mathbb{A} \mathcal{P} u\|_{p,\Omega} \lesssim \|\mathbb{A} u\|_{p,\Omega}$. The precise dependence of the constant on the domain follows by standard scaling arguments. $\square$

We next introduce the framework required to prove Theorem 4.1. We use the same Whitney coverings as in [28], which we recall for the reader's convenience. Firstly recall the Decomposition Lemma introduced in [54], that any open subset $\Omega \subset \mathbb{R}^n$ can be covered with a countable collection $\mathcal{W}_1 := \{S_j\}$ of closed dyadic cubes satisfying

(D_1) $\ell(S_j)/4 \leqslant \ell(S_l) \leqslant 4 \ell(S_j)$ if $S_j \cap S_l \neq \emptyset$,
(D_2) $\operatorname{int} S_j \cap \operatorname{int} S_l = \emptyset$ if $j \neq l$,
(D_3) $\ell(S_j) \leqslant \operatorname{dist}(S_j, \partial\Omega) \leqslant 4\sqrt{n} \ell(S_j)$ for all j,

where $\ell(Q)$ denotes the side–length of a cube Q. We henceforth assume that Ω is as in the statement of Theorem 4.1. We further consider a Whitney decomposition $\mathcal{W}_2 := \{Q_l\}$ of $\mathbb{R}^n \setminus \bar{\Omega}$, and further define $\mathcal{W}_3 := \{Q \in \mathcal{W}_2 \colon \ell(Q) \leqslant \varepsilon\delta/(16n)\}$. We reflect each cube $Q \in \mathcal{W}_3$ to a non–unique interior cube $Q^* \in \mathcal{W}_1$ such that

(R_1) $\ell(Q) \leqslant \ell(Q^*) \leqslant 4\ell(Q)$,

(R_2) $\operatorname{dist}(Q, Q^*) \leqslant C\ell(Q)$,

where above and in the following C denotes a constant depending on $k, p, n, \varepsilon, \delta$ only; additional dependencies will be specified. The non–uniqueness causes no issues, as one can show that

(R_3) For any two choices S_1, S_2 of Q^*, we have $\operatorname{dist}(S_1, S_2) \leqslant C\ell(Q)$,

(R_4) For any $S \in \mathcal{W}_1$, there are at most C cubes $Q \in \mathcal{W}_3$ such that $S = Q^*$,

(R_5) For any adjacent $Q_1, Q_2 \in \mathcal{W}_3$, we have $\operatorname{dist}(Q_1^*, Q_2^*) \leqslant C\ell(Q_1)$.

For detail on theses basic properties of the reflection see [28, Lem. 2.4-7]. We conclude the presentation of the decomposition by quoting the following:

Lemma 4.3 ([28, Lem. 2.8]). *For any adjacent cubes $Q_1, Q_2 \in \mathcal{W}_3$, there is a chain $\mathcal{C}(Q_1^*, Q_2^*) := \{Q_1^* =: S_1, S_2, \ldots S_m := Q_2^*\}$ of cubes in $S_j \in \mathcal{W}_1$, i.e., such that S_j and S_{j+1} touch for all j, and $m \leqslant C$.*

We proceed to define the extension operator

$$E_\Omega u := \begin{cases} u & \text{in } \Omega \\ \sum_{Q \in \mathcal{W}_3} \varphi_Q \pi_{Q^*} u & \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \end{cases}$$

where $\{\varphi_Q\}_{Q \in \mathcal{W}_3} \subset \mathrm{C}^\infty(\mathbb{R}^n)$ is a partition of unity such that for all $Q \in \mathcal{W}_3$ we have

(P_1) $0 \leqslant \varphi_Q \leqslant 1$ and $\sum_{Q \in \mathcal{W}_3} \varphi_Q = 1$ in $\bigcup \mathcal{W}_3$,

(P_2) $\operatorname{spt} \varphi_Q \subset 17/16 Q$, where λQ denotes the homothety of Q by λ about its centre,

(P_3) $|\nabla^l \varphi_Q| \leqslant C\ell(Q)^{-l}$ for all $0 \leqslant l \leqslant k$.

Our proof mostly follows the lines of the original proof. We first prove an estimate on chains in $\mathcal{W}_1$, then suitably bound the norms of the derivatives in the exterior domains, and we conclude by showing that the extension has weak derivatives in full–space. We warn the reader that in the remainder of this section we may use the properties of the decomposition, reflection and partition of unity without mention.

Lemma 4.4 ([28, Lem. 3.1]). *Let $\mathcal{C} := \{S_1, \ldots S_m\} \subset \mathcal{W}_1$ be a chain. Then for $0 \leqslant l < k$ we have*

$$\|\nabla^l(\pi_{S_1} u - \pi_{S_m} u)\|_{p,S_1} \leqslant C(m)\ell(S_1)^{k-l} \|\mathbb{A}u\|_{p,\cup\mathcal{C}}$$

for all $u \in \mathrm{C}^\infty(\bar{\Omega}, V)$.

Proof. We remark that L^p–norms of polynomials of degree at most d on adjacent cubes in $\mathcal{W}_1$ are comparable (see, e.g., [28, Lem. 2.1]). We get

$$\begin{aligned} \text{LHS} &\leqslant \sum_{j=1}^{m-1} \|\nabla^l(\pi_{S_{j+1}} u - \pi_{S_j} u)\|_{p,S_1} \\ &\leqslant C(m) \sum_{j=1}^{m-1} \|\nabla^l(\pi_{S_{j+1}} u - \pi_{S_j \cup S_{j+1}} u)\|_{p,S_{j+1}} + \|\nabla^l(\pi_{S_j \cup S_{j+1}} u - \pi_{S_j} u)\|_{p,S_j} \\ &\leqslant C(m) \sum_{j=1}^{m-1} \big(\|\nabla^l(\pi_{S_{j+1}} u - u)\|_{p,S_{j+1}} + 2\|\nabla^l(u - \pi_{S_j \cup S_{j+1}} u)\|_{p,S_j \cup S_{j+1}} \\ &+\|\nabla^l(u - \pi_{S_j} u)\|_{p,S_j}\big) \end{aligned}$$

and we can use the Poincaré–type inequality, Proposition 4.2, to conclude. □

Lemma 4.5 ([28, Prop. 3.4]). *For $1 \leqslant p \leqslant \infty$, we have $\|E_\Omega u\|_{\mathrm{V}^{\mathbb{A},p}(\mathbb{R}^n \setminus \bar{\Omega})} \leqslant C\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)}$ for all $u \in \mathrm{C}^\infty(\bar{\Omega}, V)$.*

Proof. We estimate on each cube in $\mathcal{W}_2$, distinguishing between small and large cubes. We also distinguish between $\mathbb{A}$ and the derivatives of order less than k. Let $Q_0 \in \mathcal{W}_3$. Then, since φ_Q sum to one in Q_0 and $\mathbb{A}\pi_{Q_0^*}u \equiv 0$, we have

$$\begin{aligned}
\|\mathbb{A}E_\Omega u\|_{p,Q_0} &= \left\| \mathbb{A} \sum_{Q\in\mathcal{W}_3} \varphi_Q(\pi_{Q^*}u - \pi_{Q_0^*}u) \right\|_{p,Q_0} \\
&\leqslant \left\| \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \mathbb{A}(\varphi_Q(\pi_{Q^*}u - \pi_{Q_0^*}u)) \right\|_{p,Q_0} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \sum_{j=0}^{k-1} \||\nabla^{k-j}\varphi_Q||\nabla^j(\pi_{Q^*}u - \pi_{Q_0^*}u)|\|_{p,Q_0} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \sum_{j=0}^{k-1} \ell(Q_0)^{j-k}\|\nabla^j(\pi_{Q^*}u - \pi_{Q_0^*}u)\|_{p,Q_0^*} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|\mathbb{A}u\|_{p,\cup\mathcal{C}(Q_0^*,Q^*)},
\end{aligned}$$

where the last inequality follows from Lemma 4.4. With a similar reasoning we obtain, for $0 \leqslant l \leqslant k-1$, that

$$\|\nabla^l E_\Omega u\|_{p,Q_0} \leqslant C\left(\|\nabla^l u\|_{p,Q_0^*} + \ell(Q_0)^{k-l} \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|\mathbb{A}u\|_{p,\cup\mathcal{C}(Q_0^*,Q^*)}\right).$$

We move on to the case $Q_0 \in \mathcal{W}_2 \setminus \mathcal{W}_3$, so if $Q \cap Q_0 \neq \emptyset$, then $\ell(Q) \geq \ell(Q_0)/4 \geq \varepsilon\delta/(64n) \geq C$. Let $0 \leqslant l \leqslant k-1$, so that

$$\begin{aligned}
\|\nabla^l E_\Omega u\|_{p,Q_0} &\leqslant \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|\nabla^l(\varphi_Q \pi_{Q^*}u)\|_{p,Q_0} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \sum_{j=1}^{l} \ell(Q_0)^{j-l}\|\nabla^j \pi_{Q^*}u\|_{p,Q_0} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \sum_{j=1}^{l} \ell(Q_0)^{j-l}\|\nabla^j \pi_{Q^*}u\|_{p,Q^*} \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \sum_{j=1}^{l} \ell(Q_0)^{j-l}(\|\nabla^j(\pi_{Q^*}u - u)\|_{p,Q^*} + \|\nabla^j u\|_{p,Q^*}) \\
&\leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|u\|_{\mathrm{V}^{l,p}(Q^*,V)} + \ell(Q_0)^{k-l}\|\mathbb{A}u\|_{p,Q^*}.
\end{aligned}$$

As, above, we similarly show that $\|\mathbb{A}E_\Omega u\|_{p,Q_0} \leqslant C\sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|u\|_{\mathrm{V}^{\mathbb{A},p}(Q^*)}$. There is no loss in assuming that $\ell(Q_0) \leqslant 1$ for any $Q_0 \in \mathcal{W}_2$, so that we can collect the estimates to obtain

$$\|E_\Omega u\|_{\mathrm{V}^{\mathbb{A},p}(Q_0)} \leqslant C \sum_{\emptyset\neq Q_0\cap Q\in\mathcal{W}_3} \|u\|_{\mathrm{V}^{\mathbb{A},p}(\mathcal{C}(Q_0^*,Q^*))}.$$

It remains to use local finiteness of the partition of unity (see, e.g., [28, Eqn. (3.1-4)]) and Lemma 5.6 to conclude. □

Proof of Theorem 4.1. We firstly show that $E_\Omega u$ has weak derivatives in $\mathbb{R}^n$, for which it suffices (by Lemma 5.5) to show that E_Ω maps $u \in \mathrm{V}^{k,\infty}(\bar{\Omega}, V)$ into $\mathrm{V}^{k,\infty}(\mathbb{R}^n, V)$. This we do in two steps. First, we show that the obvious candidate $(\nabla^l u)\chi_{\bar{\Omega}} + (\nabla^l E_\Omega u)\chi_{\mathbb{R}^n\setminus\bar{\Omega}}$

is bounded for all $0 \leqslant l \leqslant k$. We need only prove this for $l = k$, the other cases being dealt with in Lemma 4.5 for $p = \infty$. As before, we first take $Q_0 \in \mathcal{W}_3$, where

$$\begin{aligned}|\nabla^k E_\Omega u| &\leqslant |\nabla^k \pi_{Q_0^*} u| + \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} |\nabla^k (\varphi_Q (\pi_{Q^*} u - \pi_{Q_0^*} u))| \\ &\leqslant C \left(|\nabla^k \pi_{Q_0^*} u| + \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\nabla^k u\|_{\infty, \mathcal{C}(Q_0^*, Q^*)} \right).\end{aligned}$$

Clearly, $P \mapsto \|\nabla^k P\|_{\infty, Q_0^*}$ is a norm on $\mathbb{R}_d[x]^V / \mathbb{R}_{k-1}[x]$, whereas $P \mapsto \|\nabla^k \Pi P\|_{\infty, Q_0^*}$ is a semi–norm. We therefore get that $\|\nabla^k \pi_{Q_0^*} u\|_{\infty, Q_0^*} \leqslant C \|\nabla^k \mathcal{P}_{Q_0^*} u\|_{\infty, Q_0^*} \leqslant C \|\nabla^k u\|_{\infty, Q_0^*}$, where the latter inequality is given by the stability of averaged Taylor polynomials. Now consider the other case, when $Q_0 \in \mathcal{W}_2 \setminus \mathcal{W}_3$, and recall that then $\ell(Q_0) \geq C$. We have

$$\begin{aligned}|\nabla^l E_\Omega u| &\leqslant \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} |\nabla^k (\varphi_Q \pi_{Q^*})| \leqslant C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k \ell(Q)^{j-k} |\nabla^j \pi_\Omega u| \\ &\leqslant C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k \ell(Q_0)^{j-k} |\nabla^j \pi_\Omega u| \leqslant C \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \sum_{j=1}^k |\nabla^j \pi_\Omega u|,\end{aligned}$$

so we can conclude as in the previous step.

The second step is to show that $\nabla^l E_\Omega u$ is continuous for $0 \leqslant l < k$. To this end, it suffice to show that

$$\|\nabla^l E_\Omega u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} \to 0 \qquad \text{as } \ell(Q_0) \to 0$$

for $Q_0 \in \mathcal{W}_3$. Here $(\,\cdot\,)_S$ denotes the average with respect to Lebesgue measure on S. By the triangle inequality and properties of the partition of unity, we get

$$\begin{aligned}\|\nabla^l E_\Omega u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} &\leqslant \left\| \nabla^l \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \varphi_Q (\pi_{Q^*} - \pi_{Q_0^*}) \right\|_{\infty, Q_0} \\ &\quad + \|\nabla^l \pi_{Q_0^*} u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0} = \mathbf{I} + \mathbf{II}.\end{aligned}$$

By an estimation which is by now routine (see the proof of Lemma 4.5) we have that

$$\mathbf{I} \leqslant C \ell(Q_0)^{k-l} \sum_{\emptyset \neq Q_0 \cap Q \in \mathcal{W}_3} \|\mathbb{A} u\|_{\infty, \cup \mathcal{C}(Q_0^*, Q^*)},$$

which tends to zero as $\ell(Q_0) \to 0$ since $k > l$. For the second term, we have by [28, Lem. 2.1] and closeness of Q_0 and Q_0^* that

$$\begin{aligned}\mathbf{II} &\leqslant C \|\nabla^l \pi_{Q_0^*} u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0^*} \leqslant C \|\nabla^l \pi_{Q_0^*} u - \nabla^l u\|_{\infty, Q_0^*} + C \|\nabla^l u - (\nabla^l u)_{Q_0^*}\|_{\infty, Q_0^*} \\ &\leqslant C \ell(Q_0) \|\mathbb{A} u\|_{\infty, Q_0^*} + C \ell(Q_0) \|\nabla^{l+1} u\|_{\infty, Q_0^*},\end{aligned}$$

where the last inequality we used Proposition 4.2 and the fact that $\nabla^l u$ is Lipschitz as $l < k$.

We next note that by density of smooth functions in $\mathrm{W}^{\mathbb{A},p}(\Omega)$ (Lemma 5.5), it suffices to prove boundedness of the extension for maps in $u \in \mathrm{C}^\infty(\bar{\Omega}, V)$. By the argument above, we have that $E_\Omega u \in \mathrm{V}^{k,\infty}(\mathbb{R}^n, V)$, so that

$$\begin{aligned}\nabla^l E_\Omega u &= (\nabla^l u) \chi_{\bar{\Omega}} + (\nabla^l E_\Omega u) \chi_{\mathbb{R}^n \setminus \bar{\Omega}} \qquad \text{for } 0 \leqslant l \leqslant k-1 \\ \mathbb{A} E_\Omega u &= (\mathbb{A} u) \chi_{\bar{\Omega}} + (\mathbb{A} E_\Omega u) \chi_{\mathbb{R}^n \setminus \bar{\Omega}}.\end{aligned}$$

It follows that

$$\|E_\Omega u\|_{\mathrm{V}^{\mathbb{A},p}(\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{V}^{\mathbb{A},p}(\Omega)} + \|E_\Omega u\|_{\mathrm{V}^{\mathbb{A},p}(\mathbb{R}^n \setminus \bar{\Omega})} \leqslant C \|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)},$$

where in the last inequality we used Lemmas 4.5 and 5.6. The proof is complete. ∎

4.2. **Proofs of the main results.** We now begin the proof of Theorem 1.1. It is clear that (b) implies (c) and that (d) implies (e). We first prove that (a) implies (b) in full generality.

Proof of Theorem 1.3 (sufficiency of FDN). Since $\mathbb{A}$ has FDN, by Theorem 1.2, there exists a bounded, linear extension operator $E_B\colon \mathrm{W}^{\mathbb{A},1}(B) \to \mathrm{V}^{\mathbb{A},1}(\mathbb{R}^n)$. A close inspection of the proof of Theorem 4.1 reveals that E_B maps restrictions to the ball B of $\mathrm{C}^\infty(\mathbb{R}^n, V)$–functions into $\mathrm{C}_c^\infty(\tilde{B}, V)$ for a larger ball $\tilde{B} \Supset B$, which depends on B only. We write $p := n/(n-k+s)$ and use Hölder's Inequality to get that

$$
\begin{aligned}
\|u\|_{\mathrm{B}_q^{s,p}(B,V)} &\leqslant \|E_B u\|_{\mathrm{B}_q^{s,p}(\mathbb{R}^n,V)} = \|E_B u\|_{\mathrm{L}^p(\tilde{B},V)} + \|E_B u\|_{\dot{\mathrm{B}}_q^{s,p}(\mathbb{R}^n,V)} \\
&\lesssim \|E_B u\|_{\mathrm{L}^{\frac{n}{n-1}}(\tilde{B},V)} + \|E_B u\|_{\dot{\mathrm{B}}_q^{s,p}(\mathbb{R}^n,V)} \\
&\lesssim \|\nabla^{k-1} E_B u\|_{\mathrm{L}^{\frac{n}{n-1}}(\tilde{B},V)} + \|E_B u\|_{\dot{\mathrm{B}}_q^{s,p}(\mathbb{R}^n,V)}
\end{aligned}
$$

where the last estimate follows from Poincaré's Inequality with zero boundary values. We conclude by [52, Thm. 1.3, Thm. 8.4] and boundedness of E_B. $\square$

We will complete the proof of Theorem 1.3 (i.e., show necessity of FDN) at the end of this section. Returning to Theorem 1.1, to see that (b) implies (d), we prove the following:

Theorem 4.6. *Let $\mathbb{A}$ be as in (1.4) with $k = 1$. Suppose that $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(B, V)$. Then $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^q(B, V)$ for all $1 \leqslant q < \frac{n}{n-1}$.*

The proof of Theorem 4.6 relies on the Riesz-Kolmogorov criterion and the following Nikolskiĭ–type estimate:

Lemma 4.7 (Nikolskiĭ–type Estimate)**.** *Let $\mathbb{A}$ be an elliptic operator of the form (1.4), $k = 1$. Fix $R > 0$. Then for every $0 < s < 1$ there exists a constant $c = c(s, R) > 0$ such that if $u \in \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$ vanishes identically outside $B(0, R)$, then there holds*

$$
\int_{\mathbb{R}^n} |u(x+y) - u(x)|^p \,\mathrm{d}x \leqslant c \|\mathbb{A}u\|^p_{\mathrm{L}^1(B(0,R),W)} |y|^{sp}.
$$

whenever $p < n/(n-1+s)$.

Note that by Ornstein's Non–Inequality, $s = 1$ is not allowed in the lemma. A more general, sharp version, can be found in [52, Prop. 8.22].

Proof of Theorem 4.6. Recall that by the Riesz–Kolmogorov Theorem on relatively compact subsets of L^p–spaces [13, Thm. 4.26] on $\Omega \subset \mathbb{R}^n$ open and $1 \leqslant p < \infty$, a subset $\mathcal{F} \subset \mathrm{L}^p(\Omega, V)$ is relatively compact in $\mathrm{L}^p(\Omega, V)$ if and only if

(i) $\mathcal{F}$ is a bounded set in $\mathrm{L}^p(\Omega, V)$ and
(ii) for all $\varepsilon > 0$ there exists $\delta > 0$ such that there holds

$$
\|\bar{f}(\cdot + y) - \bar{f}(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n,V)} < \varepsilon, \tag{4.2}
$$

for all $f \in \mathcal{F}$ and all $y \in \mathbb{R}^n$ with $|y| < \delta$

Here $\bar{f}$ is the trivial extension of $f \in \mathcal{F}$ to $\mathbb{R}^n$.

Let $1 \leqslant q < 1^*$ and $\mathcal{F}$ be the unit ball in $\mathrm{W}^{\mathbb{A},1}(B)$. The embedding $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{n/(n-1)}(B, V)$ implies that $\mathcal{F}$ is bounded in $\mathrm{L}^q(B, V)$ which shows condition (i) of the Riesz–Kolmogorov criterion. As to (ii), let $\varepsilon > 0$ be arbitrary. Given $\varrho > 0$ sufficiently small (to be determined later on), let $\widetilde{\rho}_\varrho\colon [0, 1] \to [0, 1]$ be the Lipschitz function given by

$$
\rho_\varrho(t) := \begin{cases} 1 & \text{if } 0 < t < 1 - 2\varrho, \\ -\frac{1}{\varrho} t + \frac{1-\varrho}{\varrho} & \text{if } 1 - 2\varrho < t < 1 - \varrho, \\ 0 & \text{if } 1 - \varrho < t \leqslant 1, \end{cases}
$$

and put $\rho_\varrho(x) := \widetilde{\rho}_\varrho(|x|)$, $x \in \mathbb{R}^n$, and finally set, for given $f \in \mathcal{F}$, $f_\varrho := \rho_\varrho f$. Denoting $B_t := B(0,t)$ for $t > 0$, we note that if $|y| < \varrho$, then $f(\cdot + y) - f(\cdot)$ and $f_\varrho(\cdot + y) - f_\varrho(\cdot)$ coincide on $B_{1-3\varrho}$. Let $f \in \mathrm{W}^{\mathbb{A},1}(B)$ be arbitrary. We split

$$\int_{\mathbb{R}^n} |\bar{f}(x+y) - \bar{f}(x)|^q \,\mathrm{d}x = \left(\int_{\mathbb{R}^n \setminus B_{1-3\varrho}} + \int_{B_{1-3\varrho}} \right) |\bar{f}(x+y) - \bar{f}(x)|^q \,\mathrm{d}x =: \mathbf{I}_\varrho + \mathbf{II}_\varrho,$$

with an obvious definition of $\mathbf{I}_\varrho$ and $\mathbf{II}_\varrho$.

Ad $\mathbf{I}_\varrho$. As $|y| < \varrho$, if $x \in \mathbb{R}^n \setminus B_{1-3\varrho}$, then $x + y \in \mathbb{R}^n \setminus B_{1-4\varrho}$. Therefore, we obtain with a constant $c > 0$ independent of $f \in \mathcal{F}$

$$\begin{aligned} \mathbf{I}_\varrho \leqslant c \int_{B_1 \setminus B_{1-4\varrho}} |f(z)|^q \,\mathrm{d}x &\leqslant c \left(\int_B |f|^{\frac{n}{n-1}} \,\mathrm{d}x \right)^{\frac{(n-1)q}{n}} \mathscr{L}^n(B_1 \setminus B_{1-4\varrho})^{\frac{n-q(n-1)}{n}} \\ &\leqslant c \mathscr{L}^n(B_1 \setminus B_{1-4\varrho})^{\frac{n-q(n-1)}{n}} \end{aligned}$$

and we may hence record that there exists $\delta_1 > 0$ such that if $0 < \varrho < \delta_1$, then $\mathbf{I}_\varrho < \varepsilon/3$.

Ad $\mathbf{II}_\varrho$. Firstly, since $1 \leqslant q < n/(n-1)$, we find and fix $0 < s < 1$ such that $q < n/(n-1+s)$. By Lemma 5.8, $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{1^*}(B,V)$ implies that $\mathbb{A}$ is elliptic so that we are in position to suitably apply Lemma 4.7. Since $f(\cdot+y)-f(\cdot)$ equals $f_\varrho(\cdot+y)-f_\varrho(\cdot)$ on $B_{1-3\varrho}$ and f_ϱ is compactly supported in $\mathbb{R}^n$ with supports in a sufficiently large fixed ball, we find with a constant $c > 0$ independent of $f \in \mathcal{F}$

$$\begin{aligned} \mathbf{II}_\varrho = \int_{B_{1-3\varrho}} |f_\varrho(x+y) - f_\varrho(x)|^q \,\mathrm{d}x &\leqslant \int_{\mathbb{R}^n} |f_\varrho(x+y) - f_\varrho(x)|^q \,\mathrm{d}x \\ &\leqslant c \|\mathbb{A} f_\varrho\|_{\mathrm{L}^1(B_R,W)}^q |y|^{sq} \\ &\leqslant c \big(\|\rho_\varrho \mathbb{A} f\|_{\mathrm{L}^1(B_R,W)}^q + \|f \otimes_{\mathbb{A}} D\rho_\varrho\|_{\mathrm{L}^1(B_R,W)}^q \big) |y|^{sq} \\ &\leqslant c \big(\|\mathbb{A} f\|_{\mathrm{L}^1(B_R,W)}^q + \|f \otimes_{\mathbb{A}} D\rho_\varrho\|_{\mathrm{L}^1(B_R,W)}^q \big) |y|^{sq} \end{aligned} \tag{4.3}$$

Pick $\delta_2 > 0$ such that if $|y| < \delta_2$, then $c \sup_{f \in \mathcal{F}} \|\mathbb{A} f\|_{\mathrm{L}^1(B_R,W)}^q |y|^{sq} < \varepsilon/3$. Finally, we note because of $|D\rho_\varrho| \leqslant 4/\varrho$ by definition of ρ_ϱ,

$$\left(\int_{\mathbb{R}^n} |D\rho_\varrho|^n \,\mathrm{d}x \right)^{\frac{1}{n}} \leqslant \frac{c}{\varrho} \big((1-2\varrho)^n - (1-3\varrho)^n \big)^{\frac{1}{n}} = 1 + \mathcal{O}(\varrho) \tag{4.4}$$

and so, by $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{L}^{1^*}(B,V)$ and since $0 < \varrho \leqslant 1$,

$$\begin{aligned} \|D\rho_\varrho \otimes_{\mathbb{A}} f\|_{\mathrm{L}^1(B_R,W)}^q |y|^{sq} &\leqslant \left(\int_{\mathbb{R}^n} |D\rho_\varrho|^n \,\mathrm{d}x \right)^{\frac{q}{n}} \sup_{f \in \mathcal{F}} \|f\|_{\mathrm{W}^{\mathbb{A},1}(B)}^q |y|^{sq} \\ &\leqslant C \sup_{f \in \mathcal{F}} \|f\|_{\mathrm{W}^{\mathbb{A},1}(B)}^q |y|^{sq}, \end{aligned}$$

and from here it is evident that there exists $\delta_3 > 0$ such that if $|y| < \delta_3$, then $\|D\rho_\varrho \otimes_{\mathbb{A}} f\|_{\mathrm{L}^1(B_R,W)}^q |y|^{sq} < \varepsilon/3$ and so, by (4.3), $\mathbf{II}_\delta < 2\varepsilon/3$ for all $f \in \mathcal{F}$. Now let $0 < \delta < \varrho := \min\{\delta_1, \delta_2, \delta_3\}$. Collecting estimates, we see that (ii) is satisfied and thus we can conclude that the compact embedding $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^q(B,V)$ holds. □

With an inexpensive modification of the proof of Theorem 4.6, one can show that (c) implies that $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^q(B,V)$ for all $1 \leqslant q < p$, which trivially then implies (e).

Proof of Theorem 1.1. It remains to see that (e) implies (a), which is now a simple consequence of the Equivalence Lemma 5.2. We choose $E_1 = \mathrm{W}^{\mathbb{A},1}(B)$, $E_2 = \mathrm{L}^1(B,W)$, $E_3 = \mathrm{L}^1(B,V)$, and $A := \mathbb{A} \in \mathscr{L}(\mathrm{W}^{\mathbb{A},1}(B), \mathrm{L}^1(B,W))$, whereas $B := \iota$ is the embedding operator $\iota\colon \mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow\hookrightarrow \mathrm{L}^1(B,V)$. It is then clear that $\|u\|_{\mathrm{W}^{\mathbb{A},1}(B)} = \|u\|_*$, so the equivalence lemma yields that $\mathbb{A}$ has finite dimensional null–space. □

Proof of Theorem 1.3 (necessity of FDN). Assume that the embedding holds. By standard embeddings of Besov spaces, we have that $\mathrm{W}^{\mathbb{A},1}(B)\hookrightarrow \mathrm{W}^{k-1,p}(B,V)$ for some $p>1$. If $k=1$, we use Theorem 1.1, (c) implies (a), to see that $\mathbb{A}$ has FDN. Otherwise, we give the following simple argument: assume that $\mathbb{A}$ is not FDN, so that the maps $u_j(x)=\exp(jx\cdot\xi)v$ lie in $\ker\mathbb{A}$ for some non–zero complex ξ,v. We traced this example back to [40], but it was likely known before (cp. [3, Eq. (3.2)]). The assumed embedding and Hölder's Inequality give

$$\begin{aligned} j^{k-1}\left(\int_B|\exp(jx\cdot\xi)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} &\lesssim \|u_j\|_{\mathrm{W}^{k-1,p}(B,V)}\lesssim\|u_j\|_{\mathrm{L}^1(B,V)}\\ &\lesssim\left(\int_B|\exp(jx\cdot\xi)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}, \end{aligned}$$

which leads to a contradiction as $j\to\infty$. Here constants depend on $\operatorname{diam}B,p,n$ only. □

4.3. **Concluding Remarks.** In view of Theorem 1.3, it is quite surprising that the Sobolev–type embedding of $\mathrm{W}^{\mathbb{A},1}(B)$ holds for operators of arbitrary order whereas the techniques used to prove the trace embedding [12, Thm. 4.17–18] seem to be difficult to extend past first order operators. If $\mathbb{A}$ has FDN, we can use Theorem 1.3 to give a sub–optimal trace embedding

$$\mathrm{W}^{\mathbb{A},1}(B)\hookrightarrow\mathrm{W}^{s-\frac{1}{p},p}(\partial B,V)\qquad\text{for } s\uparrow k,\text{ so } p=\frac{n}{n-k+s}\downarrow 1, \tag{4.5}$$

using standard trace theory for Besov spaces. At this stage, a straightforward generalisation of the arguments of [12] even yields that $\mathrm{W}^{\mathbb{A},1}(B)$–maps admit traces in $\mathrm{W}^{k-1,1}(\partial B)$. However, this does not yield the sharp boundary trace space as is expected, e.g., for the gradient: The optimal embedding for $\mathbb{A}=\nabla^k$ was established only recently by Mironescu and Russ in [35], building on the $k=2$ case proved by Uspenskiĭ in [47]. They proved that the trace operator is continuous onto $B^{k-1,1}_1$, which is in general strictly smaller than $\mathrm{W}^{k-1,1}$ (see [14, Rk. A.1]). However, by modifying the arguments of [12] one might make the following conjecture:

Conjecture 4.8. *An operator $\mathbb{A}$ as in (1.4) has FDN if and only if there exists a continuous, linear, surjective trace operator* $\mathrm{Tr}\colon\ \mathrm{W}^{\mathbb{A},1}(B)\to\mathrm{B}^{k-1,1}_1(\partial B,V)$.

A few remarks are in order. Necessity of FDN can be proved by a modification of the arguments in [12, Sec. 4.3]. Surjectivity is obvious, using [35, Thm. 1.3-4] and $\mathrm{W}^{k,1}(B,V)\hookrightarrow\mathrm{W}^{\mathbb{A},1}(B)$. The difficulty stems from proving boundedness (hence, well-definedness) of the trace operator, which cannot be reduced to the situation in [35] by Ornstein's Non–Inequality, or to (4.5) by strict inclusion of Besov spaces. We do not see a straightforward way to merge the techniques in [12, 35] and intend to tackle the problem in the future.

5. Appendix

5.1. **Miscellaneous background.** The following relevant facts we quote without proof:

Lemma 5.1 ([52], Proposition 6.1). *Let $\mathbb{A}$ as in (1.4) be elliptic. Then $\mathbb{A}$ is cancelling if and only if we have that*

$$\int_{\mathbb{R}^n}\mathbb{A}u\,\mathrm{d}x=0$$

for all $u\in\mathrm{C}^\infty(\mathbb{R}^n,V)$ such that the support of $\mathbb{A}u$ is compact.

Lemma 5.2 (Peetre–Tartar Equivalence Lemma, [46, Lem. 11.1]). *Let E_1 be a Banach space and let E_2,E_3 be two normed spaces (with corresponding norms $\|\cdot\|_i$, $i\in\{1,2,3\}$) and let $A\in\mathscr{L}(E_1,E_2)$ and $B\in\mathscr{L}(E_1,E_3)$ be two bounded linear operators such that B is compact and the norms $\|\cdot\|_1$ and $\|\cdot\|_*:=\|A\cdot\|_2+\|B\cdot\|_3$ are equivalent on E_1. Then* $\dim(\ker A))<\infty$.

Theorem 5.3 ([29, Thm. 4])**.** *Let $\mathbb{A}$ as in (1.4) be $\mathbb{C}$–elliptic, and $\Omega \subset \mathbb{R}^n$ be a star–shaped domain with respect to a ball. Then there exist an integer $d := d(\mathbb{A})$, a linear map $\mathcal{P} \in \mathscr{L}(\mathrm{C}^\infty(\bar{\Omega}, V), \mathbb{R}_d[x]^V)$ and a smooth map $K \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \triangle, \mathscr{L}(W, V))$, where $\triangle = \{(x,x) \colon x \in \mathbb{R}^n\}$ such that $|D^\alpha_x D^\beta_y K(x,y)| \lesssim |x-y|^{k-n-|\alpha|-|\beta|}$ for all multi–indices α, β and all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle$, and*

$$u(x) = \mathcal{P}u(x) + \int_\Omega K(x,y)\mathbb{A}u(y)\,\mathrm{d}y$$

for all $x \in \Omega$ and $u \in \mathrm{C}^\infty(\bar{\Omega}, V)$. Therefore $\ker \mathbb{A} \subseteq \mathbb{R}_d[x]^V$.

5.2. **Other facts about $\mathrm{W}^{\mathbb{A},p}$.** We collect some complementary results that explain, e.g., our choice of definition for the $\mathbb{A}$–Sobolev spaces and of extension technique for $p = 1$.

Definition 5.4. *A connected open set $\Omega \subset \mathbb{R}^n$ is called a*

(a) C^0–domain *if for any $x \in \partial\Omega$ there exist a neighbourhood $\mathcal{N}$ of x relatively open in Ω, a coordinate system in $\mathbb{R}^n$ and a continuous function f such that, in the new coordinates (x', x_n), $\mathcal{N} = \{(x', x_n) \colon 0 < x_n < f(x'),\ x' \in B_1(0)\}$.*

(b) $\mathrm{C}^{0,1}$– *(or* Lipschitz–*)*domain *if Ω is a C^0–domain and the function f above can be chosen to be Lipschitz.*

(c) domain with the cone property *if for any $x \in \Omega$ there exists a cone $\mathcal{C}$ with apex at x and a coordinate system with respect to which, for some constants $c_i > 0$ we have $\mathcal{C} = \{(x', x_n) \colon |x'|^2 \leqslant c_1 x_n 2,\ 0 \leqslant x_n \leqslant c_2\}$.*

(d) star–shaped domain *(with respect to a ball $B \subset \Omega$) if for all $x \in \Omega$, $y \in B$, and $0 \leqslant \theta \leqslant 1$ we have that $\theta x + (1-\theta) y \in \Omega$.*

We collect a few facts from [33, Sec. 1.1] on bounded domains, which will be used without mention in the sequel: any star–shaped domain is Lipschitz; Lipschitz domains have the cone property; domains with the cone property can be written as finite unions of star–shaped domains. The following density result closely mimics [33, Sec. 1.1.4-5]. We reproduce the proof here since, on one hand, it is very elegant and, on the other, it is crucial to prove the extension Theorem 4.1.

Lemma 5.5. *Let $\mathbb{A}$ be as in (1.4), $1 \leqslant p < \infty$, and $\Omega \subset \mathbb{R}^n$ be a bounded C^0–domain. Then $\mathrm{C}^\infty(\bar{\Omega}, V)$ is dense in $\mathrm{W}^{\mathbb{A},p}(\Omega)$. The same holds true for $\mathrm{V}^{\mathbb{A},p}(\Omega)$.*

Proof. We only prove the result for $\mathrm{V}^{\mathbb{A},p}$, the other case following in the same manner.

Step I. We first show that $\mathrm{C}^\infty(\Omega, V) \cap \mathrm{V}^{\mathbb{A},p}(\Omega)$ is dense in $\mathrm{V}^{\mathbb{A},p}(\Omega)$. This step requires no regularity or boundedness assumption on Ω, other that it is open in $\mathbb{R}^n$.

Consider a Whitney decomposition $\{Q_j\}_{j=1}^\infty$ of Ω (which is locally finite), and let $\varepsilon \in (0, 1/2)$ and $\rho_j \in \mathrm{C}^\infty_c(Q_j)$ be a partition of unity associated with this decomposition. We denote by v_j a mollification of $\rho_j u$ such that

(a) $\operatorname{spt} v_j$ is also a locally finite cover of Ω;

(b) If $\operatorname{diam}\operatorname{spt} v_j = \lambda_j \operatorname{diam} Q_j$, then $\lambda_j \downarrow 0$;

(c) $\|\rho_j u - v_j\|_{\mathrm{V}^{\mathbb{A},p}(\Omega)} \leqslant \varepsilon^j$ for all $j \geq 1$.

To make (c) plain, we recall, that mollification and weak derivatives are interchangeble. It follows that $v = \sum v_j \in \mathrm{C}^\infty(\Omega, V)$ and $u = \sum \rho_j u$ in any compact subset of Ω. Moreover, due to the upper bound on ε, we obtain

$$\|u - v\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)} \leqslant \sum_{j=1}^\infty \|\rho_j u - v_j\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)} \leqslant \varepsilon(1-\varepsilon)^{-1} \leqslant 2\varepsilon,$$

which proves that $v \in \mathrm{V}^{\mathbb{A},p}(\Omega)$ and concludes the proof of this step.

Step II. To conclude the proof of the Lemma, by Step I, it suffices to show density of $\mathrm{C}^\infty(\bar{\Omega}, V)$ in $\mathrm{C}^\infty(\Omega, V) \cap \mathrm{V}^{\mathbb{A},p}(\Omega)$.

We cover the boundary of Ω by open neighbourhoods $\{\mathcal{N}_x\}_{x \in \partial\Omega}$, where each $\mathcal{N}_x$ is the graph of a continuous function as in Definition 5.4(a). We extract a finite subcolection $\{\mathcal{N}_j\}_j$ that still covers $\partial\Omega$ and let $\{\rho_j\}_j$ be a partition of unity associated with $\{\mathcal{N}_j\} \cup \mathcal{N}$, where $\Omega\setminus\bigcup\mathcal{N}_j \Subset \mathcal{N} \Subset \Omega$. Since $u=\sum \rho_j u$ in Ω, it suffices to prove the claim for u and Ω relabelled by $\rho_j u$ and $\mathcal{N}_j\cap\Omega$, respectively. In coordinates (x',x_n) as in Definition 5.4(a), we choose $u_\varepsilon(x',x_n)=u(x',x_n-\varepsilon)$ for small enough $\varepsilon>0$. Clearly $u_\varepsilon\in \mathrm{C}^\infty(\bar\Omega,V)$ and

$$\|\partial^\alpha u-\partial^\alpha u_\varepsilon\|_{\mathrm{L}^p(\Omega,V)}\to 0 \qquad \text{as } \varepsilon\downarrow 0 \text{ whenever } \partial^\alpha u\in \mathrm{L}^p(\Omega,V),$$

which completes the proof. $\square$

Lemma 5.6. *Let $\mathbb{A}$ be as in (1.4) have FDN and $\Omega\subset\mathbb{R}^n$ be a bounded Lipschitz domain. Then $\mathrm{W}^{\mathbb{A},p}(\Omega)\simeq \mathrm{V}^{\mathbb{A},p}(\Omega)$, for each $1\leqslant p\leqslant\infty$.*

Proof. One embedding is clear by definition. Conversely, we first prove the inequality under the extra assumption that Ω is star–shaped with respect to a ball. Let $u\in\mathrm{W}^{\mathbb{A},p}(B)$. We use Proposition 4.2 to estimate, for $1\leqslant l\leqslant k-1$,

$$\|\nabla^l u\|_{p,\Omega}\leqslant \|\nabla^l(u-\pi_\Omega u)\|_{p,\Omega}+\|\nabla^l\pi_\Omega u\|_{p,\Omega}\lesssim \|\mathbb{A}u\|_{p,\Omega}+\|\nabla^l\pi_\Omega u\|_{p,\Omega}$$

To estimate the latter term, we note that $P\mapsto\|\nabla^l P\|_{p,\Omega}$ defines a semi–norm on $\mathbb{R}_d[x]^V$, so it is controlled by the L^p–norm of P. We have that

$$\|\nabla^l\pi_\Omega u\|_{p,\Omega}\lesssim\|\pi_\Omega u\|_{p,\Omega}\leqslant\|u-\pi_\Omega u\|_{p,\Omega}+\|u\|_{p,\Omega}\lesssim\|\mathbb{A}u\|_{p,\Omega}+\|u\|_{p,\Omega},$$

where the last inequality follows by another application of Proposition 4.2. Altogether, we have proved that

$$\|u\|_{\mathrm{V}^{\mathbb{A},p}(\Omega)}\leqslant C(\Omega)\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)}. \tag{5.1}$$

We now assume just that Ω is Lipschitz, hence has the cone property. Hence there exist M sub–domains Ω_i that are star–shaped with respect to a ball and cover $\Omega=\bigcup_{i=1}^M\Omega_i$. We apply (5.1) in each Ω_i to get

$$\|u\|_{\mathrm{V}^{\mathbb{A},p}(\Omega)}\leqslant\sum_{i=1}^M\|u\|_{\mathrm{V}^{\mathbb{A},p}(\Omega_i)}\leqslant\sum_{i=1}^M C(\Omega_i)\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega_i)}\leqslant\max_{1\leqslant i\leqslant M}C(\Omega_i)\sum_{i=1}^M\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega_i)}.$$

Let now $1\leqslant p<\infty$. By concavity of the function $[0,\infty)\ni t\mapsto t^{1/p}$ and Jensen's Inequality, we obtain

$$\sum_{i=1}^M\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega_i)}\leqslant M^{1-1/p}\|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)}. \tag{5.2}$$

If $p=\infty$, we simply estimate $\|u\|_{\mathrm{W}^{\mathbb{A},\infty}(\Omega_i)}$ by $\|u\|_{\mathrm{W}^{\mathbb{A},\infty}(\Omega)}$ to note that (5.2) holds for $p=\infty$ as well (with the convention $1/p=\infty$). The proof is complete. $\square$

Lemma 5.7. *Let $\mathbb{A}$ as in (1.4) have FDN, $1<p<\infty$, and $\Omega\subset\mathbb{R}^n$ be a star–shaped domain with respect to a ball. Then there exists a bounded, linear extension operator $E_\Omega\colon \mathrm{W}^{\mathbb{A},p}(\Omega)\to\mathrm{V}^{k,p}(\mathbb{R}^n,V)$.*

Proof. We use the extension suggested in [30], namely, in the notation of Theorem 5.3,

$$E_\Omega u(x):=\eta(x)\left(\mathcal{P}u(x)+\int_\Omega K(x,y)\mathbb{A}u(y)\,\mathrm{d}y\right)$$

for $u\in\mathrm{C}^\infty(\bar\Omega,V)$ and $x\in\mathbb{R}^n$. Here $\eta\in\mathrm{C}_c^\infty(\mathbb{R}^n)$ is a smooth cut–off that equals 1 in a neighbourhood of Ω. We abbreviate $\mathcal{K}u=\int_\Omega K(\,\cdot\,,y)\mathbb{A}u(y)\,\mathrm{d}y$. Let $0\leqslant l\leqslant k$, and let B be a ball containing the support of η. Then, with domain dependent constants,

$$\|\nabla^l E_\Omega u\|_{p,B}\lesssim\sum_{j=0}^l\|\nabla^j(\mathcal{P}u+\mathcal{K}u)\|_{p,B}\leqslant\|\mathcal{P}u\|_{\mathrm{V}^{l,p}(B,V)}+\sum_{j=0}^l\|\nabla^j\mathcal{K}u\|_{p,B}.$$

We note that $\|\cdot\|_{\mathrm{V}^{l,p}(B,V)}$ and $\|\cdot\|_{\mathrm{L}^p(\Omega,V)}$ both define norms on $\mathbb{R}_d[x]^V$, hence they are equivalent. We also remark that $\nabla^j\mathcal{K}u=\int_\Omega\nabla_x^j K(\,\cdot\,,y)\mathbb{A}u(y)\,\mathrm{d}y$, so that

$$\|\nabla^j\mathcal{K}u\|_{p,B}\leqslant\|\nabla^j\mathcal{K}u\|_{p,\mathbb{R}^n}\lesssim\|\mathbb{A}u\|_{p,\Omega}. \tag{5.3}$$

If $0 \leqslant j < k$, the proof of (5.3) is presented in the proof of Proposition 4.2. If $j = k$, (5.3) follows from [42, Ch. II] and the growth bounds on $\nabla_x^k K$. Collecting, we get

$$\|\nabla^l E_\Omega u\|_{p,B} \lesssim \|\mathcal{P}u\|_{p,\Omega} + \|\mathbb{A}u\|_{p,\Omega} \leqslant \|\mathcal{P}u + \mathcal{K}u\|_{p,\Omega} + \|\mathcal{K}u\|_{p,\Omega} + \|\mathbb{A}u\|_{p,\Omega} \lesssim \|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)},$$

where the last inequality is obtained from (5.3) with $j = 0$. □

Lemma 5.8. *Let $\mathbb{A}$ be as in (1.4) and $\Omega \subset \mathbb{R}^n$ be a bounded, open set. If $\mathrm{W}^{\mathbb{A},1}(\Omega) \hookrightarrow \mathrm{W}^{k-1,p}(\Omega, V)$ for some $p > 1$, then $\mathbb{A}$ is elliptic.*

Proof. Suppose $\mathbb{A}$ is not elliptic, such that there exist $\xi \in \mathbb{S}^{n-1}$, $v \in V \setminus \{0\}$ such that $\mathbb{A}[\xi]v = 0$. Consider open cubes Q_1, Q_2 in $\mathbb{R}^n$ such that ξ is normal to one of their faces and $\bar{Q}_1 \subset \Omega \subset Q_2$, of side–lengths $2l_1$, $2l_2$, respectively. We put $u(x) = f((x - x_0) \cdot \xi)v$ for x_0 the centre of Q_1 and $f(t) = |t|^{k-1-1/p}$ if $t \in \mathbb{R} \setminus \{0\}$. We have that $\mathbb{A}u = 0$ and

$$\int_\Omega |u| \, \mathrm{d}x \leqslant \int_{Q_2} |u| \, \mathrm{d}x = \int_{Q_2} |f((x - x_0) \cdot \xi)||v| \, \mathrm{d}x = l_2^{n-1}|v| \int_{-l_2}^{l_2} |t|^{k-1-1/p} \, \mathrm{d}t < \infty,$$

so that $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$. On the other hand,

$$\begin{aligned} \int_\Omega |D^{k-1}u|^p \, \mathrm{d}x \geq \int_{Q_1} |D^{k-1}u|^p \, \mathrm{d}x &= \int_{Q_1} |f^{(k-1)}((x - x_0) \cdot \xi)|^p |v \otimes^{k-1} \xi|^p \, \mathrm{d}x \\ &= l_1^{n-1}|v \otimes^{k-1} \xi|^p \int_{-l_1}^{l_1} |t|^{-1} \, \mathrm{d}t = \infty, \end{aligned}$$

so that $u \notin \mathrm{W}^{k-1,p}_{\mathrm{loc}}(\Omega, V)$. The proof is complete. □

Lemma 5.9. *Let $\mathbb{A}$ be as in (1.4). If $\mathrm{W}^{\mathbb{A},1}(B) \hookrightarrow \mathrm{W}^{k-1,n/(n-1)}(B, V)$, then $\mathbb{A}$ is elliptic and cancelling.*

Proof. Necessity of ellipticity follows via Lemma 5.8. We next show that our assumed embedding implies $\dot{\mathrm{W}}^{\mathbb{A},1}(\mathbb{R}^n) \hookrightarrow \dot{\mathrm{W}}^{k-1,n/(n-1)}(\mathbb{R}^n, V)$ by a scaling argument, so that cancellation follows by the necessity part of [52, Thm. 1.3]. Let $u \in \mathrm{C}_c^\infty(\mathbb{R}^n, V)$ be such that $\operatorname{spt} u \subset B_r := B(0, r)$. Then $u_r(x) := u(rx)$ for $x \in \mathbb{R}^n$ is also a test function, with $\operatorname{spt} u_r \subset B := B(0, 1)$. We estimate, with constants independent of r:

$$\begin{aligned} \|D^{k-1}u\|_{\mathrm{L}^{\frac{n}{n-1}}} &= \left(\int_{B_r} |D^{k-1}u(x)|^{\frac{n}{n-1}} \, \mathrm{d}x \right)^{\frac{n-1}{n}} = \left(\int_B r^{\frac{n(k-1)}{n-1}} |D^{k-1}u_r(y)|^{\frac{n}{n-1}} r^n \, \mathrm{d}y \right)^{\frac{n-1}{n}} \\ &= r^{n-k} \left(\int_B |D^{k-1}u_r(y)|^{\frac{n}{n-1}} \, \mathrm{d}y \right)^{\frac{n-1}{n}} \leqslant c r^{n-k} \int_B |\mathbb{A}u_r(y)| + |u_r(y)| \, \mathrm{d}y \\ &= c \int_{B_r} |\mathbb{A}u(x)| \, \mathrm{d}x + c r^{-k} \int_{B_r} |u(x)| \, \mathrm{d}x \leqslant c \int_{B_r} |\mathbb{A}u(x)| \, \mathrm{d}x = c\|\mathbb{A}u\|_{\mathrm{L}^1}, \end{aligned}$$

where the last inequality follows from a change of variable and the Poincaré–type inequality with zero boundary values (for elliptic operators)

$$\|v\|_{\mathrm{L}^1(\Omega, V)} \leqslant c(\operatorname{diam} \Omega)^k \|\mathbb{A}v\|_{\mathrm{L}^1(\Omega, W)} \tag{5.4}$$

for all $v \in \mathrm{C}_c^\infty(\Omega, V)$. The proof is complete. □

The inequality (5.4) follows from an iteration of Poincaré's Inequality, the Green–type Formula (2.1), and Theorem 2.2(b), in the following way:

$$\begin{aligned} \|v\|_{\mathrm{L}^1(\Omega)} &\leqslant c(\operatorname{diam} \Omega)^{k-1} \|D^{k-1}v\|_{\mathrm{L}^1(\Omega)} = c(\operatorname{diam} \Omega)^{k-1} \|\mathbb{K}_{\mathbb{A}} \star (\mathbb{A}v)\|_{\mathrm{L}^1(\Omega)} \\ &\leqslant c(\operatorname{diam} \Omega)^{k-1} \|I_1|\mathbb{A}v|\|_{\mathrm{L}^1(\Omega)} \leqslant c(\operatorname{diam} \Omega)^k \|\mathbb{A}v\|_{\mathrm{L}^1(\Omega)} \end{aligned}$$

A similar, straightforward argument also gives the inequality in Lemma 5.10 below.

Lemma 5.10. *Let $\mathbb{A}$ as in (1.4) be elliptic, $k = 1$. Then for each $1 \leqslant p < n/(n-1)$, there exists $c > 0$ such that*

$$\|u\|_{\mathrm{L}^p(B,V)} \leqslant c\|\mathbb{A}u\|_{\mathrm{L}^1(B,W)}$$

for all $u \in \mathrm{C}_c^\infty(B, V)$.

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