FA Revision

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Functional Analysis Revision Class

Problem 1: Terminology

Give examples of

(a) Banach spaces which are not Hilbert,

(b) Banach spaces which are not separable,

(c) Banach spaces on which not any two norms are equivalent,

(d) a linear operator T on some Banach space X which is bounded, but discontinuous,

(e) a linear operator $T\colon X \to Y$ between two Banach spaces X, Y which is bounded and continuous,

(f) a linear operator $T\colon X \to Y$ between two Banach spaces X, Y which is not bounded.

Moreover, decide whether the following are true or false – if they are false, give the correct statement:

(g) For any $1 \leq p \leq \infty$ and any $f \in \mathrm{L}^p(\mathbb{R}^n)$, there exists $(f_j) \subset \mathrm{C}_c^\infty(\mathbb{R}^n)$ such that $f_j \to f$ in $\mathrm{L}^p(\mathbb{R}^n)$.

(h) For any $1 \leq p \leq \infty$ we have $\mathrm{L}^p(\Omega)^* \cong \mathrm{L}^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, $1' = \infty$, $\infty' = 1$.

Solution. Ad 1. This is the case, e.g., for any L^p-space for $p \in [1,\infty] \setminus \{2\}$. Note carefully that a necessary condition for a norm to stem from an inner product is the parallelogram identity / polarisation identity. Ad 2. Non-separable Banach space are given by $\mathrm{L}^\infty((0,1))$ or ℓ^∞. Ad 3. Any infinite dimensional Banach space will do. Ad 4. Such an operator does not exist – for linear operator, boundedness is equivalent to continuity. Ad 5. Any bounded linear operator is continuous – so we may take the identity on $X = Y$, for instance. Ad 6. Take $X = \mathrm{C}([0,1])$ and consider $W := \mathrm{C}^1([0,1])$, $Y = \mathrm{C}^1([0,1])$ and equip all the spaces with the usual supremum norm. Then the derivative operator $\frac{\mathrm{d}}{\mathrm{d}x}\colon W \ni f \mapsto \frac{\mathrm{d}}{\mathrm{d}x} f \in \mathrm{C}([0,1])$ is a linear operator. However, it is not bounded: Consider $f_j(x) := \frac{1}{j}\sin(jx)$, which satisfies $\|f_j\|_\infty \to 0$ but $\|f_j'\|_\infty = 1$. Now note carefully that $(\mathrm{C}^1([0,1]), \|\cdot\|_\infty)$ is *not* (!) Banach. To overcome this issue, take an algebraic complement $\widetilde{W}$ of W in X, i.e., $X = W \oplus \widetilde{W}$ and extend the derivative operator by zero (on $\widetilde{W}$) to the entire X. The operator that arises in this way then yields an example for 6.. Ad 7. Yes for $1 \leq p < \infty$ – which is a classical result from Analysis 3 (keywords: cut-off and mollification) – but no for $p = \infty$. Recall the argument: Pick some $u \in (\mathrm{L}^\infty \setminus \mathrm{C})(\mathbb{R}^n)$. Suppose such a sequence $(f_j) \subset \mathrm{C}_c^\infty(\mathbb{R}^n)$ exists. Then (f_j) is Cauchy for $\|\cdot\|_{\mathrm{L}^\infty}$, but on $\mathrm{C}(\mathbb{R}^n)$ the L^∞-norm equals the supremum norm. Hence (f_j) converges to some element in the closure of $\mathrm{C}_c^\infty(\mathbb{R}^n)$ for $\|\cdot\|_{\sup}$, which is $\mathrm{C}_0(\mathbb{R}^n)$. This would imply that $u \in \mathrm{C}_0(\mathbb{R}^n)$, a contradiction. Ad 8. This was discussed in class – yes for $1 \leq p < \infty$, no for $p = \infty$. ∎

Problem 2:

Decide with proof which of the following are proper dense subspaces of $\ell^2(\mathbb{N})$:

$$\mathfrak{A} := \{x = (x_j) \in \ell^2(\mathbb{N}) \colon\ x_{2019} \geq 0\},$$
$$\mathfrak{B} := \{x = (x_j) \in \ell^2(\mathbb{N}) \colon\ x_{2019} = 0\},$$
$$\mathfrak{C} := \{x = (x_j) \in \ell^2(\mathbb{N}) \colon\ \sum_j |\sin(x_j)| < \infty\}.$$

Solution. Given in class. ■

Problem 3:

Decide with proof whether the following subsests of $\ell^2(\mathbb{N})$ are bounded and/or precompact and/or compact:

$$\mathcal{A} := \{x = (x_j) \colon\ \|x\|_{\ell^2} \leq 1\},$$
$$\mathcal{B} := \{x = (x_j) \colon\ |x_j| \leq \frac{1}{\sqrt{j}} \text{ for all } j \in \mathbb{N}\},$$
$$\mathcal{C} := \{x = (x_j) \colon\ |x_j| \leq \frac{1}{j} \text{ for all } j \in \mathbb{N}\}.$$

Solution. Given in class. ■