# BOUNDARY ELLIPTICITY AND LIMITING L¹-ESTIMATES ON HALFSPACES 

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$$
\begin{aligned}
& \text { AbSTRACT. We identify necessary and sufficient conditions on } k \text { th order dif- } \\
& \text { ferential operators } \mathbb{A} \text { in terms of a fixed halfspace } H^{+} \subset \mathbb{R}^{n} \text { such that the } \\
& \text { Gagliardo-Nirenberg-Sobolev inequality } \\
& \qquad\left\|D^{k-1} u\right\|_{L^{\frac{n}{n-1}}\left(H^{+}\right)} \leqslant c\|\mathbb{A} u\|_{L^{1}\left(H^{+}\right)} \text {for } u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)
\end{aligned}
$$

holds. This comes as a consequence of sharp trace theorems on $H=\partial H^{+}$.

## 1. Introduction

Let $\mathbb{A}$ be a homogeneous, linear, vectorial $k$ th order partial differential operator with constant coefficients between the finite dimensional real inner product spaces $V$ and $W$. That is, $\mathbb{A}$ has a representation

$$
\begin{equation*}
\mathbb{A} u(x)=\sum_{|\alpha|=k} A_{\alpha} \partial^{\alpha} u(x), \quad u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right) \tag{1.1}
\end{equation*}
$$

where $A_{\alpha} \in \operatorname{Lin}(V, W)$ for $\alpha \in \mathbb{N}_{0}^{n},|\alpha|=k$. In this context, a classical question is the validity of the estimate

$$
\begin{equation*}
\left\|D^{k} u\right\|_{\mathrm{L}^{p}\left(H^{+}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{p}\left(H^{+}\right)} \quad \text { for } u \in \mathrm{C}_{c}^{\infty}\left(\overline{H^{+}}, V\right) \tag{1.2}
\end{equation*}
$$

for some given open halfspace $H^{+} \subset \mathbb{R}^{n}$ with a constant $c>0$ independent of $u$. A positive answer implies that replacing in the Sobolev norm the full derivative $D^{k} u$ by $\mathbb{A} u$ is an equivalent norm on the Sobolev space, which is well adapted to problems in the Calculus of Variations and partial differential equations involving $\mathbb{A}$; most notably, such coercive inequalities lead to well-posedness theorems in non-linear elasticity or fluid mechanics, see e.g. [20, 21, 29]. By standard techniques such as flattening the boundary, one can then reduce the corresponding estimates on smooth domains and for operators with variable coefficients to the halfspace case.

When $1<p<\infty$, the cases where (1.2) holds can be characterized following Aronszajn [4, Thm. V], in terms of the symbol of $\mathbb{A}$ defined as

$$
\mathbb{A}(\xi)=\sum_{|\alpha|=k} \xi^{\alpha} A_{\alpha}, \quad \text { for } \xi \in \mathbb{R}^{n}
$$

as follows: Estimate (1.2) holds if and only if both of the following conditions hold:
(a) $\mathbb{A}$ is elliptic (in the interior), i.e., $\operatorname{ker}_{\mathbb{R}} \mathbb{A}(\xi)=\{0\}$ for all $\xi \in \mathbb{R}^{n} \backslash\{0\} ;$
(b) $\mathbb{A}$ is boundary elliptic, by which we mean that we have $\operatorname{ker}_{\mathbb{C}} \mathbb{A}(\xi+\mathrm{i} \nu)=\{0\}$ for all $\xi \in \mathbb{R}^{n}$; here, $\nu$ denotes a unit normal to the hyperplane $H:=\partial H^{+}$.
Different from the Calderón-Zygmund theory [11, 12] for $1<p<\infty$, Ornstein has shown that when $p=1$, then (1.2) does not hold unless one has the trivial pointwise estimate $\left|D^{k} u(x)\right| \leqslant c|\mathbb{A} u(x)|[35,28,19]$. This obstruction is also referred to as Ornstein's non-inequality and we emphasize that there is no effect of the boundary

[^0]here: The condition is necessary for (1.2) to hold for compactly supported functions. Similar results hold for $p=\infty[34,15]$.

Consequently, strong $L^{1}$ estimates, if possible, must bound weaker derivatives. In the case of interior estimates, building on the fundamental work of BourgainBrezis [6, 7, 50] and its higher-order generalization [51], it was shown in [53] that a Gagliardo-Nirenberg inequality

$$
\begin{equation*}
\left\|D^{k-1} u\right\|_{\mathrm{L}^{\frac{n}{n-1}}\left(\mathbb{R}^{n}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n}\right)} \quad \text { for } u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right) \tag{1.3}
\end{equation*}
$$

holds if and only if the operator $\mathbb{A}$ is elliptic and the equation $\mathbb{A} u=\delta_{0} w_{0}$ has no solution for $w_{0} \in W \backslash\{0\}$. This additional assumption, termed cancellation, can be expressed algebraically as

$$
\begin{equation*}
\bigcap_{\xi \in \mathbb{S}^{n-1}} \operatorname{im} \mathbb{A}(\xi)=\{0\} \tag{C}
\end{equation*}
$$

via the Fourier transform. This cancellation condition plays an important role in endpoint estimates of Hardy type and into Lorentz, Besov and Triebel-Lizorkin spaces, see for instance $[40,39,41,45,25,44,16,46,52]$.

When $1<p<\infty$, the condition on the boundary (b) is a Lopatinskiü-Shapiro or covering condition [31]. Such conditions were used successfully by Agmon-DouglisNirenberg and Hörmander, among many others, to provide a satisfactory theory for determined and overdetermined elliptic systems [2, 3, 27].

When $p=1$, estimates were established from different perspectives in [14], and in [10] for Poisson's equation with divergence free data. In [23], it was shown that if $\Omega \subset \mathbb{R}^{n}$ is a bounded domain with a smooth boundary $\partial \Omega$, then the inequality

$$
\left\|D^{k-1} u\right\|_{\mathrm{L}^{\frac{n}{n-1}}(\Omega)} \leqslant c\left(\|\mathbb{A} u\|_{\mathrm{L}^{1}(\Omega)}+\|u\|_{\mathrm{L}^{1}(\Omega)}\right) \quad \text { for } u \in \mathrm{C}^{\infty}(\bar{\Omega}, V)
$$

is equivalent with boundary ellipticity of $\mathbb{A}$ in all directions,

$$
\operatorname{ker}_{\mathbb{C}} \mathbb{A}(\xi)=\{0\} \quad \text { for all } \xi \in \mathbb{C}^{n} \backslash\{0\}
$$

We say that these operators are $\mathbb{C}$-elliptic. This condition also plays a crucial role in establishing trace estimates in Lebesgue or Besov spaces [9, 17, 24]. However, little is known towards a comprehensive theory of global estimates for elliptic boundary value problems with $\mathrm{L}^{1}$ data.

In view of the above discussion, the present paper gives a complete answer to the question of proving the Sobolev analogue of Aronszajn's result (1.2) for $p=1$. In the following, we denote for $\nu \in \mathbb{S}^{n-1}$

$$
\begin{equation*}
H_{\nu}:=\left\{x \in \mathbb{R}^{n}: x \cdot \nu=0\right\} \quad \text { and } \quad H_{\nu}^{ \pm}:=\left\{x \in \mathbb{R}^{n}: \operatorname{sgn}(x \cdot \nu)= \pm 1\right\} \tag{1.4}
\end{equation*}
$$

the hyperplane with normal $\nu$ together with the corresponding adjacent halfspaces, and we note that in this terminology, $\nu$ is the inward unit normal to $\partial H_{\nu}^{+}$. The above classification problem is solved by the following theorem, displaying the first main result of the present paper:

Theorem 1.1. Let $n \geq 2$ and $\mathbb{A}$ be a kth order differential operator as in (1.1). Then the following are equivalent:
(a) The operator $\mathbb{A}$ is elliptic (i.e., $\operatorname{ker}_{\mathbb{R}} \mathbb{A}(\xi)=\{0\}$ for $\xi \in \mathbb{R}^{n} \backslash\{0\}$ ) and boundary elliptic in direction $\nu \in \mathbb{S}^{n-1}$ (i.e., $\operatorname{ker}_{\mathbb{C}} \mathbb{A}(\xi+\mathrm{i} \nu)=\{0\}$ for all $\left.\xi \in \mathbb{R}^{n}\right)$.
(b) There exists a constant $c>0$ such that the Sobolev estimate

$$
\left\|D^{k-1} u\right\|_{\mathrm{L}^{\frac{n}{n-1}}\left(H_{\nu}^{+}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)}
$$

holds for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
To prove Theorem 1.1, we establish sharp trace theorems on given halfspaces which enable us to extend to full space. Recall that the trace space of $\dot{\mathrm{W}}^{k, 1}\left(H_{\nu}^{+}\right)$is $\mathrm{L}^{1}\left(H_{\nu}\right)$
[22] for $k=1$ and the Besov space $\dot{\mathrm{B}}_{1,1}^{k-1}\left(H_{\nu}\right)$ for $k \geq 2$ [49]. It is therefore natural to split our analysis into first and higher order operators. For $k=1$, we have:

Theorem 1.2. Let $n \geq 2$ and $\mathbb{A}$ be a differential operator as in (1.1) with $k=1$. Then the following are equivalent:
(a) The operator $\mathbb{A}$ is boundary elliptic in direction $\nu \in \mathbb{S}^{n-1}$.
(b) There exists a constant $c>0$ such that the trace estimate

$$
\begin{equation*}
\|u\|_{\mathrm{L}^{1}\left(H_{\nu}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \tag{1.5}
\end{equation*}
$$

holds for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
Even for operators with simple structure, estimating differences to arrive at (1.5) a la Gagliardo [22] turns out to be hard as beyond the usual gradients this has only been achieved, to the best of our knowledge, for symmetric gradients [47, 5]. As for all other examples in which trace estimates are known [9, 17, 24], the methods make instrumental use the fact that the differential operators concerned are boundary elliptic in every direction. Therefore, there is no hope to make these approaches work in the sharp case of Theorem 1.2.

Our proof of the trace inequality in Theorem 1.2 uses an improved version of Smith's integral representation formula from [43], see Theorem 2.6 below. Crucially using the homogeneity and regularity properties of the underlying integral kernels, this brings us in a position to employ a similar argument as in Gagliardo's original proof of the trace inequality for $\mathrm{W}^{1,1}$-maps [22], see Proposition 3.1(a). This $\mathrm{L}^{1}$-estimate, however, is insufficient to prove the optimal higher order trace inequalities, which require Besov space estimates; see Proposition 3.1(b). In this regard, we will prove the following:

Theorem 1.3. Let $n \geq 2$ and $\mathbb{A}$ be a differential operator as in (1.1) of order $k \geq 2$. Then the following are equivalent:
(a) The operator $\mathbb{A}$ is boundary elliptic in direction $\nu \in \mathbb{S}^{n-1}$.
(b) There exists a constant $c>0$ such that the estimate

$$
\left\|\partial_{\nu}^{k-1} u\right\|_{\mathrm{L}^{1}\left(H_{\nu}\right)}+\sum_{j=0}^{k-2}\left\|\partial_{\nu}^{j} u\right\|_{\dot{\mathrm{B}}_{1,1}^{k-1-j}\left(H_{\nu}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)}
$$

holds for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
The proof of this result hinges on refined Besov estimates on the integral kernels derived in Section 2. This not only yields the sharp trace theorem for boundary elliptic operators but also displays a new method for the usual $k$ th order gradient case. Moreover, from a conceptual perspective of limiting $L^{1}$-estimates involving differential operators, the proof of Theorem 1.3 seems to be the first approach that systematically uses difference estimates despite the lack of the $L^{1}$-control of full $k$ th order gradients due to Ornstein's non-inequality.

Note that in fact Theorems 1.2 and 1.3 lead to a unified treatment of first and higher order operators by considering the space

$$
\mathrm{T}_{k}\left(H_{\nu}, V\right):=\left\{\left(f_{0}, f_{1}, \ldots, f_{k-1}\right): \begin{array}{c}
f_{k-1} \in \mathrm{~L}^{1}\left(H_{\nu}, V\right) \\
f_{j} \in \dot{\mathrm{~B}}_{1,1}^{k-1-j}\left(H_{\nu}, V\right), 0 \leqslant j \leqslant k-2
\end{array}\right\}
$$

endowed with the obvious norm, cf. Theorem 4.1. We will show in Section 6 that for $k$ th order operators $\mathbb{A}$, the boundary ellipticity condition in direction $\nu$ is equivalent to the fact that $\operatorname{tr}_{k}\left(\mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)\right)=\mathrm{T}_{k}\left(H_{\nu}, V\right)$, where the trace operator is defined as

$$
\operatorname{tr}_{k} u:=\left.\left(u, \partial_{\nu} u, \ldots, \partial_{\nu}^{k-1} u\right)\right|_{H_{\nu}}
$$

for functions smooth up to the boundary and then extended by continuity for a suitable sort of strict convergence; see (6.2)ff. for the underlying terminology of such generalized BV-type spaces. We show that this trace map admits a continuous right
inverse that cannot be linear, generalizing Peetre's Theorem, cf. [36, 37, 38]. These facts seem to have gone unnoticed also in the basic case of $\mathrm{BV}^{k}\left(\mathbb{R}_{+}^{n}\right)$ and $\dot{\mathrm{W}}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)$.

To prove Theorem 1.1, we use the unified extension theorem in $\dot{\mathrm{W}}^{k, 1}\left(H_{\nu}^{-}\right)$(see Theorem 2.2) to reduce the question to an estimate in full space, in this case (1.3). To see if this estimate is available, we should combine the canceling condition and boundary ellipticity. Interestingly, it turns out that the boundary ellipticity of (a) implies the canceling condition (C), see Proposition 5.1.

Using the same extension procedure, we can prove versions up to the boundary of other estimates that are known in full space [53, 8, 40], see Theorem 5.2. This particularly allows us to bound all fractional derivatives of $D^{k-1} u$ (e.g. in the SobolevSlobodeckiĭ scale) up to, but not including, order one on halfspaces against $\mathbb{A} u$. In light of Ornstein's negative result, this displays the optimal generalisation of Aronszajn's result to the case $p=1$.

Finally, let us point out that all of the preceding theorems admit interpretations in the potential theory for elliptic systems on halfspaces with $\mathrm{L}^{1}$-data; Theorem 1.1 then corresponds to the case of elliptic systems with identically null boundary conditions. Whereas the focus of the present paper is on the generalisation of Aronszajn's result to the case $p=1$, it may also be seen as a first step towards a comprehensive theory of $\mathrm{L}^{1}$-estimates for general boundary value problems. We will pursue this in later work.

The paper is organized as follows: in Section 2 we give a comprehensive trace and extension theory in $\dot{\mathrm{W}}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)$ and establish the improvement of the representation formula from Smith's work that will be instrumental for the main results. In Section 3 we prove the main trace estimates on convolution operators. In Section 4 we prove both trace theorems, and in Section 5 we establish the Sobolev estimate and its extensions. In Section 6 we extend our estimates to spaces of rough functions and comment on the notion of boundary ellipticity.

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## 2. Traces for $\mathrm{W}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)$-maps and REpresentation formulas

In this section, we revisit the trace theory for functions in the Sobolev space $\mathrm{W}^{k, 1}$ on halfspaces and give an improved variant of Smith's representation formula to play a crucial role in the subsequent sections.

In view of our main results, we will assume that we work in space dimensions $n>1$ throughout. The reason for this is that, for $n=1$, the only relevant operator is $\mathbb{A}(t)=t^{k}$, in which case we have the embedding $\dot{\mathrm{W}}^{k, 1}\left(\mathbb{R}_{+}\right) \hookrightarrow \mathrm{C}_{0}^{k-1}\left(\mathbb{R}_{+}\right)$, where the latter space is endowed with the norm $u \mapsto\left\|u^{(k-1)}\right\|_{\infty}$. In this case, Theorem 1.1 holds with the convention $1 / 0=\infty$ and all derivatives up to order $k-1$ have well defined point values at 0 .
2.1. Traces for the Sobolev space $\mathrm{W}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)$. The results below necessitate some background facts from Besov space theory, and we refer the reader to Triebel's encyclopedic monograph [48, §5] for the definition and elementary properties of homogeneous Besov spaces. Most importantly for us, we require a characterisation of homogeneous Besov spaces in terms of finite differences [48, §5.2.3, Thm. 1] that we record explicitely: Given $k \in \mathbb{N}$ and a map $u: \mathbb{R}^{m} \rightarrow V$, we put for $h \in \mathbb{R}^{m}$

$$
\Delta_{h}^{k} u(x):=\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} u(x+i h), \quad x \in \mathbb{R}^{m}
$$

Moreover, given $s>0$ and $k \in \mathbb{N}$ with $k>s$, we define the seminorms

$$
\begin{equation*}
\|u\|_{\dot{\mathrm{B}}_{1,1}^{s}\left(\mathbb{R}^{m}\right)}:=\int_{\mathbb{R}^{m}} \int_{\mathbb{R}^{m}}\left|\Delta_{h}^{k} u(x)\right| \mathrm{d} x \frac{\mathrm{~d} h}{|h|^{m+s}} \tag{2.1}
\end{equation*}
$$

for $u \in \mathrm{~L}_{\mathrm{loc}}^{1}\left(\mathbb{R}^{m}, V\right)$, and it is clear that (2.1) defines a norm on $\mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{m}, V\right)$. We note that for any two such choices of $k>s$ the corresponding seminorms on the right-hand side of (2.1) are equivalent and, in particular, define equivalent norms on $\dot{\mathrm{B}}_{1,1}^{s}\left(\mathbb{R}^{m}, V\right)$. Upon tacitly identifying boundaries $H_{\nu}$ of half-spaces $H_{\nu}^{ \pm}$with $\mathbb{R}^{n-1}$, all of the preceding notions canonically carry over to functions defined on $H_{\nu}$.

Based on these definitions and identifying $\mathbb{R}^{n-1} \simeq \mathbb{R}^{n-1} \times\{0\}$, the classical results of Gagliardo [22] and Uspenskiĭ [49] (also see [33, 32]) can be stated as follows:

Theorem 2.1. Let $k \geq 2$. Then we have the trace inequalities

$$
\|u\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)} \leqslant c\|D u\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)} \quad \text { and } \quad\|u\|_{\dot{\mathrm{B}}_{1,1}^{k-1}\left(\mathbb{R}^{n-1}\right)} \leqslant c\left\|D^{k} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)}
$$

for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$.
As a consequence, one can immediately prove the estimate of Theorem 1.3 in the case of Sobolev spaces, $\mathbb{A}=D^{k}$ :

$$
\left\|\operatorname{tr}_{k} u\right\|_{\mathrm{T}_{k}\left(\mathbb{R}^{n-1}\right)} \leqslant c\left\|D^{k} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)} \quad \text { for all } u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)
$$

where

$$
\operatorname{tr}_{k} u:=\left.\left(u, \partial_{n} u, \ldots, \partial_{n}^{k-1} u\right)\right|_{\mathbb{R}^{n-1}}
$$

The trace operators that can be defined by the estimates of Theorem 2.1 admit continuous right inverses, but these are insufficient for our purposes. We will prove the following extension theorem, which is probably known to the experts but seems absent from the literature:

Theorem 2.2. There exists a constant $c>0$ such that for all $g_{0}, \ldots, g_{k-1} \in$ $\mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n-1}\right)$, there exists $u \in \mathrm{C}^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ such that for $j=0, \ldots, k-1$

$$
\partial_{n}^{j} u(\cdot, 0)=g_{j}
$$

and

$$
\left\|D^{k} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n}\right)} \leqslant c\left(\left\|g_{0}\right\|_{\dot{\mathrm{B}}_{1,1}^{k-1}\left(\mathbb{R}^{n-1}\right)}+\cdots+\left\|g_{k-2}\right\|_{\dot{\mathrm{B}}_{1,1}^{1}\left(\mathbb{R}^{n-1}\right)}+\left\|g_{k-1}\right\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)}\right) .
$$

The proof of this result is done in two steps: First we construct extension operators that satisfy each Dirichlet condition separately. Then we use a superposition formula to put these extensions together.

Proposition 2.3. For every $j \in\{0, \ldots, k-2\}$, if $g_{j} \in \dot{\mathrm{~B}}_{1,1}^{k-j}\left(\mathbb{R}^{n-1}\right)$, there exists $u \in \mathrm{C}^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ such that $\partial_{n}^{j} u(\cdot, 0)=g_{j}$ and

$$
\left\|D^{k} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n}\right)} \leqslant c\left\|g_{j}\right\|_{\dot{\mathrm{B}}_{1,1}^{k-j-1}\left(\mathbb{R}^{n-1}\right)}
$$

Proof. By [49] and [33, Thm. 1.2], there exists $v \in \mathrm{C}^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ such that $v(\cdot, 0)=g$ and if $\ell \leqslant j$,

$$
\int_{\mathbb{R}_{+}^{n}} x_{n}^{j-k}\left|D^{k-\ell} v\left(x^{\prime}, x_{n}\right)\right| \mathrm{d} x \leqslant c\left\|g_{j}\right\|_{\mathrm{B}_{1,1}^{k-j-1}\left(\mathbb{R}^{n-1}\right)}
$$

Defining $u\left(x^{\prime}, x_{n}\right)=x_{n}^{j} v\left(x^{\prime}, x_{n}\right) / j$ !, we reach the conclusion.
Proposition 2.4 ([37, Lem. 3.3]). If $g \in \mathrm{~L}^{1}\left(\mathbb{R}^{n-1}\right)$, there exists $u \in \mathrm{C}^{\infty}\left(\mathbb{R}_{+}^{n}\right)$ such that $\partial_{n}^{k-1} u(\cdot, 0)=g$ and

$$
\left\|D^{k} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n}\right)} \leqslant c\|g\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)}
$$

Proof. The proof given by Mironescu [32] for $p=1$ has a straightforward adaptation. Taking a function $\theta \in \mathrm{C}_{c}^{\infty}(\mathbb{R})$ such that $\theta(0)=1$ and $\theta^{\prime}(0)=\cdots=\theta^{(k-1)}(0)=0$, we define for $\varepsilon>0$ to be chosen later

$$
u\left(x^{\prime}, x_{n}\right)=\theta\left(x_{n} / \varepsilon\right) x_{n}^{k-1} g\left(x^{\prime}\right)
$$

We have

$$
\int_{\mathbb{R}_{+}^{n}}\left|D^{k} u\right| \mathrm{d} x \leqslant c \sum_{j=0}^{k} \int_{\mathbb{R}^{n-1}} \varepsilon^{k}\left|D^{k} g\right| \mathrm{d} x^{\prime} ;
$$

taking $\varepsilon>0$ small enough, we reach the conclusion.
Proof of Proposition 2.2. By Propositions 2.3 and 2.4, let $u_{j}$ be given so that

$$
\partial_{n}^{j} u_{j}(\cdot, 0)=g_{j} .
$$

We apply now a linear superposition of dilations (see [30, Thm. 2.2] and [1, Thm. 4.26]). Defining now

$$
u\left(x^{\prime}, x_{n}\right)=\sum_{j=0}^{k-1} \sum_{i=1}^{k} \mu_{i, j} u_{j}\left(x^{\prime}, \lambda_{i} x_{n}\right)
$$

with fixed distinct $\lambda_{1}, \ldots, \lambda_{k} \in(0, \infty)$ under the condition on $\mu_{i, j}$ that

$$
\sum_{i=1}^{k} \mu_{i, j} \lambda_{i}^{\ell}=\delta_{j, \ell},
$$

we reach the conclusion.
Finally, we remark that the trace operators in $\mathrm{W}^{k, 1}$, as defined by Theorem 2.1, have continuous inverses, but these can only be linear for $k>1$ [36, 33]. A generalization of Peetre's result that Gagliardo's trace operator $\operatorname{tr}\left(\mathrm{W}^{1,1}\right)=\mathrm{L}^{1}$ cannot have a bounded linear inverse to the $k$ th order Sobolev space is proved in [38, Thm. 5.1]. Here we prove the following related result:

Proposition 2.5. The bounded linear trace operator

$$
\operatorname{tr}_{k}: \dot{\mathrm{W}}^{k, 1}\left(\mathbb{R}_{+}^{n}\right) \rightarrow \operatorname{tr}_{k}\left(\dot{\mathrm{~W}}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)\right)=\mathrm{T}_{k}\left(\mathbb{R}^{n-1}\right)
$$

does not admit a right inverse that is both linear and continuous.
Proof. Suppose that $E$ is a bounded linear inverse of $\operatorname{tr}_{k}$ and let $f \in \mathrm{~L}^{1}\left(\mathbb{R}^{n-1}\right)$, so $(0, \ldots, 0, f) \in \mathrm{T}_{k}\left(\mathbb{R}^{n-1}\right)$. Write $u:=E(0, \ldots, 0, f) \in \dot{\mathrm{W}}^{k, 1}\left(\mathbb{R}_{+}^{n}\right)$ so that $\partial_{n}^{k-1} u \in$ $\dot{\mathrm{W}}^{1,1}\left(\mathbb{R}_{+}^{n}\right)$. Note that then $f \mapsto \partial_{n}^{k-1} u$ is a bounded linear inverse of Gagliardo's trace operator. This contradicts Peetre's theorem.
2.2. The Smith integral representation. In this section we revisit and improve Smith's construction of representation formulas implied by the boundary ellipticity condition [42, 43]. Precisely, we have

Theorem 2.6. Let $\mathbb{A}$ as in (1.1) be boundary elliptic in direction $\nu \in \mathbb{S}^{n-1}$. Then there exists a $(k-n)$-homogeneous convolution kernel $K \in \mathrm{C}^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}, \operatorname{Lin}(W, V)\right)$ such that $K=0$ in $H_{\nu}^{-}$and

$$
\begin{equation*}
u(x)=\int_{H_{\nu}^{+}} K(y) \mathbb{A} u(x+y) \mathrm{d} y \quad \text { for } x \in \mathbb{R}^{n} \tag{2.2}
\end{equation*}
$$

for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
We give a direct proof of Theorem 2.6, following the approach in [43, Sec. 3], where our presentation yields $\mathrm{C}^{\infty}$ - instead of $\mathrm{C}^{l}$-smoothness for fixed arbitrarily large $l \in \mathbb{N}$. We start with a variant of the Sobolev integral representations in the spirit of [43, Sec. 2]; here, we use the notation $V \otimes \otimes^{k} \mathbb{R}^{n}$ to denote the space of $V$-valued $k$-linear maps on $\mathbb{R}^{n}$.

Proposition 2.7. There exists a $(k-n)$-homogeneous convolution kernel $K_{k} \in$ $\mathrm{C}^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}, \operatorname{Lin}\left(V \otimes \bigotimes^{k} \mathbb{R}^{n}, V\right)\right)$ such that for every $x=\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n}$ with $x_{n} \leqslant|x| / 2$, $K_{k}(x)=0$ and

$$
\begin{equation*}
u(x)=\int_{\mathbb{R}_{+}^{n}} K_{k}(y) D^{k} u(x+y) \mathrm{d} y, \quad \text { for } x \in \mathbb{R}^{n} \tag{2.3}
\end{equation*}
$$

for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
Moreover, for every $r \in \mathbb{N}$, there exists $a(k+r-n)$-homogeneous convolution kernel $K_{k}^{r} \in\left(\mathbb{R}^{n} \backslash\{0\}, \operatorname{Lin}\left(V \otimes \bigotimes^{k} \mathbb{R}^{n} \otimes \otimes^{r} \mathbb{R}^{n-1}, V\right)\right)$ such that

$$
\int_{\mathbb{R}^{n-1}} K_{k}\left(y^{\prime}, y_{n}\right) v\left(y^{\prime}\right) \mathrm{d} y^{\prime}=\int_{\mathbb{R}_{+}^{n}} K_{k}^{r}\left(y, y_{n}\right) D^{r} v\left(y^{\prime}\right) \mathrm{d} y
$$

for every $v \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n-1}, V \otimes \otimes^{k} \mathbb{R}^{n}\right)$ and $y_{n} \in(0,+\infty)$.
Proof. By integration by parts we have that for $\theta \in \mathbb{S}^{n-1}$ it holds that

$$
u(x)=c \int_{0}^{\infty} t^{k-1} \frac{\mathrm{~d}^{k}}{\mathrm{~d} t^{k}} u(x+t \theta) \mathrm{d} t=c \int_{0}^{\infty} t^{k-1} D^{k} u(x+t \theta)\left[\theta^{\otimes k}\right] \mathrm{d} t
$$

We fix $\varphi \in \mathrm{C}^{\infty}([-1,1])$ such that $\varphi=0$ on $[-1,1 / 2]$ and $\int_{\mathbb{S}^{n-1}} \varphi\left(\theta_{n}\right) \mathrm{d} \theta=1$. We then have

$$
\begin{aligned}
u(x) & =c \int_{\mathbb{S}^{n-1}} \int_{0}^{\infty} t^{k-1} D^{k} u(x+t \theta)\left[\theta^{\otimes k}\right] \varphi(\theta n) \mathrm{d} t \mathrm{~d} \theta \\
& =c \int_{\mathbb{R}_{+}^{n}}|y|^{k-1} D^{k} u(x+y)\left[\left(\frac{y}{|y|}\right)^{\otimes k}\right] \varphi\left(\frac{y_{n}}{|y|}\right) \frac{\mathrm{d} y}{|y|^{n-1}}
\end{aligned}
$$

which suffices to conclude the proof of the first statement.
The second statement stems from the fact that if $H(z)=z^{\otimes m} \eta(|z|), z \in \mathbb{R}^{n-1} \times$ $\{0\} \simeq \mathbb{R}^{n-1}$, then $\operatorname{div} H=(n+m) z^{\otimes m-1} \eta(|z|)+z^{\otimes m} \eta^{\prime}(|z|)$. The kernels $K_{k}^{r}$ can then be computed recursively through the latter identity and have the required properties.
Proof of Theorem 2.6. There is no loss of generality in setting $\nu=e_{n}$, i.e., $H_{\nu}=\mathbb{R}^{n-1}$ and $H_{\nu}^{+}=\mathbb{R}_{+}^{n}$. We will use coordinates $x=\left(x^{\prime}, x_{n}\right)$ (real or complex), defined in an obvious way.

We begin by assuming that $\operatorname{dim} V=1$, so that the boundary ellipticity assumption then reduces to the condition $\mathbb{A}(\xi+i \nu) \neq 0$ for every $\xi \in \mathbb{R}^{n}$. In particular, we have

$$
\left\{\xi=\left(\xi^{\prime}, \xi_{n}\right) \in \mathbb{C}^{n}: \mathbb{A}(\xi)=0 \text { and } \xi^{\prime}=0\right\}=\{0\}
$$

therefore by Hilbert's strong Nullstellensatz (see for example [13, Chpt. 4, §2, Thm. $6]$ ), for $d \in \mathbb{N}$ large enough, there exist $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$ homogeneous differential operators of orders $d-k$ and $d-\ell$, such that

$$
\xi^{\otimes d}=\mathbb{M}_{1}(\xi) \mathbb{A}(\xi)+\mathbb{M}_{2}(\xi) \xi^{\prime \otimes \ell}
$$

By the Sobolev representation formula (Proposition 2.7) and by integration by parts, it follows that

$$
u(x)=\int_{\mathbb{R}_{+}^{n}} K_{d}(y) D^{d} u(x+y) \mathrm{d} y
$$

$$
\begin{align*}
& =\int_{\mathbb{R}_{+}^{n}} K_{d}(y) \mathbb{M}_{1} \mathbb{A} u(x+y) \mathrm{d} y+\int_{\mathbb{R}_{+}^{n}} K_{d}(y) \mathbb{M}_{2} D^{\prime \ell} u(x+y) \mathrm{d} y  \tag{2.4}\\
& =\int_{\mathbb{R}_{+}^{n}}\left(\mathbb{M}_{1}^{*} K_{d}^{*}\right)^{*}(y) \mathbb{A} u(x+y) \mathrm{d} y+(-1)^{\ell} \int_{\mathbb{R}_{+}^{n}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}(y) u(x+y) \mathrm{d} y,
\end{align*}
$$

where $K_{d}^{*}(y):=\left(K_{d}(y)\right)^{*}$ is the pointwise adjoint. We observe that operator $\left(\mathbb{M}_{1}^{*} K_{d}^{*}\right)^{*}$ is homogeneous of degree $k-n$ and that $\left(\mathbb{M}_{1}^{*} K_{d}^{*}\right)^{*}(x)=0$ if $x_{n} \leqslant|x| / 2$, so that the first term in the right-hand side of (2.4) has the required structure.

Since the operator $\mathbb{A}$ is boundary elliptic, we have $w_{0}=\mathbb{A}(0,1) \neq 0$. By homogeneity of the operator $\mathbb{A}$, it follows that for every $\xi^{\prime} \in \mathbb{R}^{n-1}, w_{0} \cdot \mathbb{A}\left(\xi^{\prime}, \cdot\right)$ is a polynomial of degree $k$, its leading order term being $\left|w_{0}\right|^{2} \xi_{n}^{k}$.

We define $\mathbb{P}\left(\xi^{\prime}\right)$ as a differential operator from $\mathbb{R}$ to the space $\mathbb{R}[W]_{k}$ of homogeneous polynomials on $W$ of degree $k$ by $\mathbb{P}\left(\xi^{\prime}\right)[w]=\operatorname{Res}\left(w_{0} \cdot \mathbb{A}(\xi), w \cdot \mathbb{A}(\xi), \xi_{n}\right)$, the resultant of the polynomials $w_{0} \cdot \mathbb{A}(\xi)$ and $w \cdot \mathbb{A}(\xi)$, seen as polynomials in $\xi_{n}$ over the ring of polynomials in $\left(\xi^{\prime}, w\right)$ (see for example [13, Chpt. 3, $\left.\S \S 5-6\right]$ ). Given $\xi^{\prime} \in \mathbb{R}^{n-1}$, we let $\tau_{1}, \ldots, \tau_{k} \in \mathbb{C}$ be the roots of the polynomial $w_{0} \cdot \mathbb{A}\left(\xi^{\prime}, \cdot\right)$ and we define the linear subspaces

$$
W_{0}=\left\{w \in W: \operatorname{deg}\left(w \cdot \mathbb{A}\left(\xi^{\prime}, \cdot\right)\right)<k\right\}
$$

and

$$
W_{j}=\left\{w \in W: w \cdot \mathbb{A}\left(\xi^{\prime}, \tau_{j}\right)=0\right\}, \quad j \in\{1, \ldots, k\}
$$

By boundary ellipticity, we have $W_{j} \neq W$ for every $j \in\{0, \ldots, k\}$. By the properties of resultants, we have

$$
\left\{w \in W: \mathbb{P}\left(\xi^{\prime}\right)[w]=0\right\} \subseteq \bigcup_{j=0}^{k} W_{j} \neq W
$$

and thus $\mathbb{P}\left(\xi^{\prime}\right) \neq 0$. Since $\mathbb{P}$ is a differential operator on scalar functions, this is equivalent to having $\mathbb{P}$ elliptic.

From the definition of the resultant $\mathbb{P}$ as a Sylvester determinant, $\mathbb{P}$ is homogeneous of degree $k^{2}$ and there exist homogeneous differential operator $\mathbb{Q}$ of order $k(k-1)$ from $W$ into $\mathbb{R}[W]_{k}$ such that

$$
\mathbb{P}\left(\xi^{\prime}\right)=\mathbb{Q}(\xi) \mathbb{A}(\xi)
$$

By ellipticity, the operator $\mathbb{P}$ on $\mathbb{R}^{n-1}$ defined by $\mathbb{P}\left(\xi^{\prime}\right)$ has a fundamental solution $E \in \mathrm{C}^{\infty}\left(\mathbb{R}^{n-1} \backslash\{0\}, \operatorname{Lin}\left(\mathbb{R}[W]_{k}, \mathbb{R}\right)\right)$, i.e.,

$$
\begin{equation*}
v\left(x^{\prime}\right)=\int_{\mathbb{R}^{n-1}} E\left(y^{\prime}\right) \mathbb{P} v\left(x^{\prime}-y^{\prime}\right) \mathrm{d} y^{\prime}, \quad \text { for } x^{\prime} \in \mathbb{R}^{n-1}, v \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n-1}\right) \tag{2.5}
\end{equation*}
$$

which is such that $D^{s} E$ is $\left(k^{2}-(n-1)-s\right)$-homogeneous for $s \in \mathbb{N}$ provided that either $n=2$ (when one is inverting a differential operator on $\mathbb{R}^{n-1}$ by the fundamental theorem of calculus) or when $s \geq k^{2}-(n-1)+1$ (see, for instance [26, Chpt. VII]).

We rewrite the second term of the right-hand side of (2.4) thanks to (2.5) as

$$
\begin{align*}
\int_{\mathbb{R}^{n}} & \left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}(y-x) u(y) \mathrm{d} y \\
& =\int_{\mathbb{R}^{n}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}(y-x)\left(\int_{\mathbb{R}^{n-1}} E\left(y^{\prime}-z^{\prime}\right) \mathbb{P} u\left(z^{\prime}, y_{n}\right) \mathrm{d} z^{\prime}\right) \mathrm{d}\left(y^{\prime}, y_{n}\right)  \tag{2.6}\\
& =\int_{\mathbb{R}^{n}}\left(\int_{\mathbb{R}^{n-1}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}(y-x) E\left(y^{\prime}-z^{\prime}\right) \mathrm{d} y^{\prime}\right) \mathbb{P} u\left(z^{\prime}, y_{n}\right) \mathrm{d}\left(z^{\prime}, y_{n}\right) \\
& =\int_{\mathbb{R}^{n}} F\left(x^{\prime}-z^{\prime}, y_{n}-x_{n}\right) \mathbb{P} u\left(z^{\prime}, y_{n}\right) \mathrm{d}\left(z^{\prime}, y_{n}\right)
\end{align*}
$$

where

$$
\begin{aligned}
F\left(y^{\prime}, y_{n}\right) & =\int_{\mathbb{R}^{n-1}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}\left(v^{\prime}-y^{\prime}, y_{n}\right) E\left(v^{\prime}\right) \mathrm{d} y^{\prime} \\
& =\int_{\mathbb{R}^{n-1}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}\left(w^{\prime}, y_{n}\right) E\left(w^{\prime}+y^{\prime}\right) \mathrm{d} w^{\prime}
\end{aligned}
$$

Since $\left(\mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}\left(w^{\prime}, y_{n}\right)=0$ whenever $\left|w^{\prime}\right| \geq 2 y_{n}$ and since $\mathbb{B} E$ is locally integrable provided $\ell \leqslant k^{2}-1$. If we assume moreover, that we have chosen $\ell=k^{2}-1$, then $D^{\prime \ell} E$ is homogeneous of degree $2-n$ and thus, by a suitable integration by parts, we get that $F$ is homogeneous of degree $k-n$.

We take $\varphi \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)$ such that $\varphi=1$ in a neighbourhood of 0 and $\varphi=0$ outside the ball of radius $\left|x^{\prime}\right| / 2$. In view of (2.6), we have that, for $\left(y^{\prime}, y_{n}\right)$ near $\left(x^{\prime}, 0\right)$, for every $r \in \mathbb{N}$,

$$
\begin{align*}
F\left(y^{\prime}, y_{n}\right)= & \int_{\mathbb{R}^{n-1}}\left(\operatorname{div}^{\prime \ell} \mathbb{M}_{2}^{*} K_{d}^{*}\right)^{*}\left(w^{\prime}, y_{n}\right) \varphi\left(w^{\prime}+y^{\prime}\right) E\left(w^{\prime}+y^{\prime}\right) \mathrm{d} w^{\prime}  \tag{2.7}\\
& +\int_{\mathbb{R}^{n-1}} K_{d}^{r}\left(w^{\prime}, y_{n}\right) D^{\prime r} \mathbb{M}_{2}^{*} D^{\prime \ell}[(1-\varphi) E]\left(w^{\prime}+y^{\prime}\right) \mathrm{d} w^{\prime}
\end{align*}
$$

The first integral defines a function that is of class $\mathrm{C}^{\infty}$ in a neighbourhood of $\left(x^{\prime}, 0\right)$; the second integral can be differentiated $d+r-1$ times without destroying the integrability, it thus follows $F$ is of class $\mathrm{C}^{d+r-1}$ in a neighbourhood of $\left(x^{\prime}, 0\right)$ with arbitrary $r \in \mathbb{N}$.

Taking $K=\left(\mathbb{M}_{1}^{*} K_{d}^{*}\right)^{*}+F$, we reach the conclusion when $\operatorname{dim} V=1$.
If $\operatorname{dim} V=m \geq 2$, we consider the operator $\wedge^{m} \mathbb{A}$ of order $m k$ from $\wedge^{m} V$ to $\wedge^{m} W$ defined for $v_{1}, \ldots, v_{m} \in V$ by

$$
\wedge^{m} \mathbb{A}(\xi)\left(v_{1} \wedge \cdots \wedge v_{m}\right)=\left(\mathbb{A}(\xi) v_{1}\right) \wedge \cdots \wedge\left(\mathbb{A}(\xi) v_{m}\right)
$$

There exists an operator $\mathbb{S}$ of order $(m-1) k$ from $W$ to $\operatorname{Lin}\left(\bigwedge^{m-1} V, \wedge^{m} W\right)$ such that for every $\omega \in \wedge^{m-1} V$,

$$
\wedge^{m} \mathbb{A}(\omega \wedge v)=(\mathbb{S A} v)(\omega)
$$

Moreover, letting $v_{1}, \ldots, v_{m}$ be a basis of $V$ and choosing $\omega_{1}, \ldots, \omega_{m} \in \wedge^{m-1} V$ such that $\omega_{i} \wedge v_{j}=\delta_{i j}$, so that for every $v \in V$, one has

$$
v=\sum_{j=1}^{m}\left(\omega_{i} \wedge v\right) v_{i}
$$

(upon identification between $\Lambda^{m} V$ and $\mathbb{R}$ ). Letting $\hat{K}$ be the homogeneous kernel of order $m k-n$ given for $\Lambda^{m} \mathbb{A}$ in the first part of the proof, we have the identity

$$
\begin{aligned}
u(x) & =\sum_{i=1}^{m}\left(\omega_{i} \wedge u(x)\right) v_{i} \\
& =\sum_{i=1}^{m} \int_{\mathbb{R}_{+}^{n}} K(y) \wedge^{m} \mathbb{A}\left(\omega_{i} \wedge u\right)(x+y) \mathrm{d} y v_{i} \\
& =\sum_{i=1}^{m} \int_{\mathbb{R}_{+}^{n}} K(y) \mathbb{S A} u(x+y)\left(\omega_{i}\right) \mathrm{d} y v_{i} \\
& =\int_{\mathbb{R}_{+}^{n}}\left(\sum_{i=1}^{m} \mathbb{S}\left[\omega_{i}\right]^{*} K(y) v_{i}\right) \mathbb{A} u(x+y) \mathrm{d} y
\end{aligned}
$$

which is the conclusion.

## 3. Estimates on special convolution operators

The main analytical advancement of this paper is contained in the following estimate for convolution operators with kernels that vanish on a halfspace; in the sequel, we identify $\mathbb{R}^{n-1} \times\{0\} \simeq \mathbb{R}^{n-1}$.

Proposition 3.1. Let $s \geq 1$ and $K \in \mathrm{C}^{\infty}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ be $(s-n)$-homogeneous and satisfy $K \equiv 0$ in $\mathbb{R}_{-}^{n}$. Let

$$
\mathcal{T} f(x):=\int_{\mathbb{R}_{+}^{n}} K(y) f(x+y) \mathrm{d} y \quad \text { for } f \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)
$$

Then we have the estimates:
(a) if $s=1$

$$
\|\mathcal{T} f\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)} \leqslant c\|f\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)} \quad \text { for } f \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right),
$$

(b) if $s>1$

$$
\|\mathcal{T} f\|_{\dot{\mathrm{B}}_{1,1}^{s-1}\left(\mathbb{R}^{n-1}\right)} \leqslant c\|f\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)} \quad \text { for } f \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}\right)
$$

We will use coordinates $x=\left(x^{\prime}, t\right), x^{\prime} \in \mathbb{R}^{n-1}$. The proof consists by first noticing that it suffices to show that $K(\cdot, 1) \in \mathrm{L}^{1}\left(\right.$ resp. $\left.\dot{\mathrm{B}}_{1,1}^{s-1}\right)$ if $s=1$ (resp. $s>1$ ), followed by checking these claims using the fact that the smoothness and vanishing of $K$ on $\mathbb{R}_{-}^{n}$ give us better bounds than homogeneity alone.

Proof. We begin with the proof of (a), the case $s=1$. We first show that it suffices to prove that $K(\cdot, 1) \in \mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)$.

We have that by a simple change of variable

$$
\begin{aligned}
\mathcal{T} f\left(x^{\prime}, 0\right) & =\int_{\mathbb{R}_{+}^{n}} K(y) f\left(\left(x^{\prime}, 0\right)+y\right) \mathrm{d} y=\int_{\left(x^{\prime}, 0\right)+\mathbb{R}_{+}^{n}} K\left(z-\left(x^{\prime}, 0\right)\right) f(z) \mathrm{d} z \\
& =\int_{\mathbb{R}_{+}^{n}} K\left(z-\left(x^{\prime}, 0\right)\right) f(z) \mathrm{d} z
\end{aligned}
$$

so that by Fubini's theorem

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}}\left|\mathcal{T} f\left(x^{\prime}, 0\right)\right| \mathrm{d} x^{\prime} & \leqslant \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}_{+}^{n}}\left|K\left(z-\left(x^{\prime}, 0\right)\right)\right||f(z)| \mathrm{d} z \mathrm{~d} x^{\prime} \\
& =\int_{\mathbb{R}_{+}^{n}}\left(\int_{\mathbb{R}^{n-1}}\left|K\left(z-\left(x^{\prime}, 0\right)\right)\right| \mathrm{d} x^{\prime}\right)|f(z)| \mathrm{d} z
\end{aligned}
$$

We will show that the inner integral is independent of $z=\left(z^{\prime}, t\right) \in \mathbb{R}_{+}^{n}$. We make the change of variable $y^{\prime}=t^{-1}\left(z^{\prime}-x^{\prime}\right)$ to get

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}}\left|K\left(z-\left(x^{\prime}, 0\right)\right)\right| \mathrm{d} x^{\prime} & =\int_{\mathbb{R}^{n-1}}\left|K\left(z^{\prime}-x^{\prime}, t\right)\right| \mathrm{d} x^{\prime} \\
& =\int_{\mathbb{R}^{n-1}} t^{1-n}\left|K\left(t^{-1}\left(z^{\prime}-x^{\prime}\right), 1\right)\right| \mathrm{d} x^{\prime}=\int_{\mathbb{R}^{n-1}}\left|K\left(y^{\prime}, 1\right)\right| \mathrm{d} y^{\prime}
\end{aligned}
$$

Therefore

$$
\begin{equation*}
\int_{\mathbb{R}^{n-1}}\left|\mathcal{T} f\left(x^{\prime}, 0\right)\right| \mathrm{d} x^{\prime} \leqslant \int_{\mathbb{R}^{n-1}}\left|K\left(y^{\prime}, 1\right)\right| \mathrm{d} y^{\prime} \int_{\mathbb{R}_{+}^{n}}|f(z)| \mathrm{d} z \tag{3.1}
\end{equation*}
$$

To show that (3.1) implies the estimate in (a), we will prove that $K(\cdot, 1) \in \mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)$.
To achieve this, we fix $0<\alpha<1$ and will only use the fact that $K \in \mathrm{C}_{\text {loc }}^{0, \alpha}\left(\mathbb{R}^{n} \backslash\{0\}\right)$ together with a homogeneity argument. We write $\mathbb{S}^{n-2}=\mathbb{R}^{n-1} \cap \mathbb{S}^{n-1}$ and denote, for an arbitrary but fixed $0<r<1$, the neighbourhood $\mathcal{N}:=\mathbb{S}^{n-2}+B_{r}(0)$ of $\mathbb{S}^{n-2}$ in $\mathbb{R}^{n}$. We then define $c_{1}$ to be the $\alpha$-Hölder seminorm of $K$ on $\mathcal{N}$, i.e.,

$$
c_{1}:=\sup _{\substack{x, y \in \mathcal{N}, x \neq y}} \frac{|K(x)-K(y)|}{|x-y|^{\alpha}} .
$$

The geometric argument depicted in Figure 1 shows that there exists a constant $c_{2}>0$ such that

$$
\begin{equation*}
\left|y^{\prime}\right|>c_{2} \Rightarrow \frac{\left(y^{\prime}, 1\right)}{\left|\left(y^{\prime}, 1\right)\right|} \in \mathcal{N} . \tag{3.2}
\end{equation*}
$$

From the same picture we see that the orthogonal projections of such points $\frac{\left(y^{\prime}, 1\right)}{\left|\left(y^{\prime}, 1\right)\right|}$ onto $\mathbb{R}^{n-1}$ are contained in $\mathcal{N}$, so

$$
\begin{equation*}
\left|y^{\prime}\right|>c_{2} \Rightarrow \frac{\left(y^{\prime}, 0\right)}{\left|\left(y^{\prime}, 1\right)\right|} \in \mathcal{N} \tag{3.3}
\end{equation*}
$$



Figure 1. The geometric argument underlying (3.2) in the proof of Theorem 1.1. When $\left|y^{\prime}\right|>c_{2}$, so $\left(y^{\prime}, 1\right) \in \mathcal{M}:=\left\{\left(x^{\prime}, 1\right):\left|x^{\prime}\right|>c_{2}\right\}$, then $\pi\left(y^{\prime}, 1\right)$, the projection of $\left(y^{\prime}, 1\right)$ onto $\mathbb{S}^{n-1}$, belongs to $\mathcal{N}$.

Therefore, if $\left|y^{\prime}\right|>c_{2}$, then (3.2), (3.3), and the $(1-n)$-homogeneity of $K$ allow us to conclude for points $\left(y^{\prime}, 1\right)$ that

$$
\begin{aligned}
\left|K\left(y^{\prime}, 1\right)\right| & =\left|\left(y^{\prime}, 1\right)\right|^{1-n}\left|K\left(\frac{\left(y^{\prime}, 1\right)}{\left|\left(y^{\prime}, 1\right)\right|}\right)\right| \\
& =\left|\left(y^{\prime}, 1\right)\right|^{1-n}\left|K\left(\frac{\left(y^{\prime}, 1\right)}{\left|\left(y^{\prime}, 1\right)\right|}\right)-K\left(\frac{\left(y^{\prime}, 0\right)}{\left|\left(y^{\prime}, 1\right)\right|}\right)\right| \\
& \leqslant c_{1}\left|\left(y^{\prime}, 1\right)\right|^{1-n}\left|\frac{\left(y^{\prime}, 1\right)}{\left|\left(y^{\prime}, 1\right)\right|}-\frac{\left(y^{\prime}, 0\right)}{\left|\left(y^{\prime}, 1\right)\right|}\right|^{\alpha}=c_{1}\left|\left(y^{\prime}, 1\right)\right|^{1-n-\alpha}
\end{aligned}
$$

where in the second equality we used the fact that $K=0$ on $\mathbb{R}^{n-1} \cap \mathcal{N}$. Then, writing $c_{3}:=\max \left\{\left|K\left(y^{\prime}, 1\right)\right|:\left|y^{\prime}\right| \leqslant c_{2}\right\}$, we have that

$$
\begin{aligned}
\int_{\mathbb{R}^{n-1}}\left|K\left(y^{\prime}, 1\right)\right| \mathrm{d} y^{\prime} & \leqslant \int_{\left\{\left|y^{\prime}\right| \leqslant c_{2}\right\}}\left|K\left(y^{\prime}, 1\right)\right| \mathrm{d} y^{\prime}+c_{1} \int_{\left\{\left|y^{\prime}\right|>c_{2}\right\}} \frac{\mathrm{d} y^{\prime}}{\left|y^{\prime}\right|^{n-1+\alpha}} \\
& =\omega_{n-1} c_{2}^{n-1} c_{3}+c_{1} \int_{c_{2}}^{\infty} \frac{r^{n-2} \mathrm{~d} r}{r^{n-1+\alpha}}=\omega_{n-1} c_{2}^{n-1} c_{3}+\frac{c_{1}}{\alpha c_{2}^{\alpha}}
\end{aligned}
$$

where $\omega_{n-1}$ denotes the ( $n-1$ )-dimensional Lebesgue measure of the ( $n-1$ )-dimensional unit ball. We conclude from (3.1) that

$$
\|\mathcal{T} f\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)} \leqslant\|K(\cdot, 1)\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)}\|f\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)}
$$

so the estimate in (a) follows.
The proof of (b), i.e., for $s>1$, follows the same structure but is more subtle. We first show that it suffices to prove that $K(\cdot, 1) \in \dot{\mathrm{B}}_{1,1}^{s-1}$.

Let $k:=\lfloor s\rfloor+1$ (or any integer larger than $s$ ) so

$$
\Delta_{h}^{k} \mathcal{T} f\left(x^{\prime}, 0\right)=\int_{\mathbb{R}_{+}^{n}} \Delta_{h}^{k} K\left(x^{\prime}+y^{\prime}, t\right) f\left(y^{\prime}, t\right) \mathrm{d} y^{\prime} \mathrm{d} t
$$

where the finite difference acts only on the $x^{\prime}$-component. We estimate

$$
\left|\Delta_{h}^{k} \mathcal{T} f\left(x^{\prime}, 0\right)\left(x^{\prime}, 0\right)\right| \leqslant \int_{\mathbb{R}_{+}^{n}}\left|\Delta_{h}^{k} K\left(x^{\prime}+y^{\prime}, t\right)\right|\left|f\left(y^{\prime}, t\right)\right| \mathrm{d} y^{\prime} \mathrm{d} t
$$

By Fubini's theorem, we have

$$
\begin{aligned}
\|\mathcal{T} f(\cdot, 0)\|_{\dot{\mathrm{B}}_{1,1}^{s-1}\left(\mathbb{R}^{n-1}\right)} & \leqslant \int_{\mathbb{R}_{+}^{n}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}}\left|\Delta_{h}^{k} K\left(x^{\prime}+y^{\prime}, t\right)\right| \frac{\mathrm{d} x^{\prime} \mathrm{d} h}{|h|^{n+s-2}}\left|f\left(y^{\prime}, t\right)\right| \mathrm{d} y^{\prime} \mathrm{d} t \\
& =\int_{\mathbb{R}_{+}^{n}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}}\left|\Delta_{h}^{k} K\left(x^{\prime}, 1\right)\right| \frac{\mathrm{d} x^{\prime} \mathrm{d} h}{|h|^{n+s-2}}\left|f\left(y^{\prime}, t\right)\right| \mathrm{d} y^{\prime} \mathrm{d} t \\
& =\|K(\cdot, 1)\|_{\dot{\mathrm{B}}_{1,1}^{s-1}\left(\mathbb{R}^{n-1}\right)}\|f\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)}
\end{aligned}
$$

where the first equality follows by a simple change of variable and the homogeneity of the kernel $K$. It remains to show that $K(\cdot, 1) \in \dot{\mathrm{B}}_{1,1}^{s-1}$, i.e., that

$$
\int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}}\left|\Delta_{h}^{k} K\left(x^{\prime}, 1\right)\right| \frac{\mathrm{d} x^{\prime} \mathrm{d} h}{|h|^{n+s-2}}<\infty
$$

To simplify the proof, we will endow $\mathbb{R}^{n}$ with the $\ell^{1}$-norm. Since $K \in C^{k}\left(\mathbb{R}^{n} \backslash\{0\}\right)$, we have for every $x=\left(x^{\prime}, t\right) \in \mathbb{R}^{n}$ with $|x|=1$ and $h \in \mathbb{R}^{n}$ such that $|h| \leqslant 1 /(2 k)$,

$$
\begin{equation*}
\left|\Delta_{h}^{k} K\left(x^{\prime}, t\right)\right| \leqslant c_{1}|h|^{k} \tag{3.4}
\end{equation*}
$$

By homogeneity of $K$, for every $x^{\prime}, h \in \mathbb{R}^{n-1}$ with $|h| \leqslant\left(1+\left|x^{\prime}\right|\right) /(2 k)$, letting $t=1 /\left(1+\left|x^{\prime}\right|\right)$, we have $\left|\left(t x^{\prime}, t\right)\right|=1$ and thus by (3.4)

$$
\left|\Delta_{h}^{k} K\left(x^{\prime}, 1\right)\right|=t^{n-s}\left|\Delta_{t h}^{k} K\left(t x^{\prime}, t\right)\right| \leqslant c_{1} t^{n-s}|t h|^{k}=\frac{c_{1}|h|^{k}}{\left(1+\left|x^{\prime}\right|\right)^{n+k-s}}
$$

Hence, we have

$$
\begin{align*}
\int_{|h| \leqslant 1 /(2 k)} & \int_{\mathbb{R}^{n-1}} \frac{\left|\Delta_{h}^{k} K\left(x^{\prime}, 1\right)\right|}{|h|^{n+s-2}} \mathrm{~d} x^{\prime} \mathrm{d} h \\
& \leqslant \int_{|h| \leqslant 1 /(2 k)} \int_{\mathbb{R}^{n-1}} \frac{c_{1}}{|h|^{n+s-2-k}\left(1+\left|x^{\prime}\right|\right)^{n+k-s}} \mathrm{~d} x^{\prime} \mathrm{d} h<\infty \tag{3.5}
\end{align*}
$$

Next, since $K=0$ on $\mathbb{R}_{-}^{n}$, we have for every $x=\left(x^{\prime}, t\right) \in \mathbb{R}^{n}$ such that $|x|=1$,

$$
\begin{equation*}
\left|K\left(x^{\prime}, t\right)\right| \leqslant c_{2}|t|^{k} . \tag{3.6}
\end{equation*}
$$

Letting $t=1 /(1+|x|)$, we have $\left|\left(t x^{\prime}, t\right)\right|=1$ and thus by (3.6)

$$
\left|K\left(x^{\prime}, 1\right)\right|=t^{n-s}\left|K\left(t x^{\prime}, t\right)\right| \leqslant c_{2} t^{n+k-s}=\frac{c_{2}}{\left(1+\left|x^{\prime}\right|\right)^{n+k-s}}
$$

and thus, since $k \geq 2$,

$$
\begin{align*}
\int_{|h| \geq 1 /(2 k)} & \int_{\mathbb{R}^{n-1}} \frac{\left|\Delta_{h}^{k} K\left(x^{\prime}, 1\right)\right|}{|h|^{n+s-2}} \mathrm{~d} x^{\prime} \mathrm{d} h \\
& \leqslant \sum_{i=0}^{k}\binom{k}{i} \int_{|h| \geq 1 /(2 k)} \int_{\mathbb{R}^{n-1}} \frac{\left|K\left(x^{\prime}+i h, 1\right)\right|}{|h|^{n+s-2}} \mathrm{~d} x^{\prime} \mathrm{d} h  \tag{3.7}\\
& \leqslant 2^{k} \int_{|h| \geq 1 /(2 k)} \int_{\mathbb{R}^{n-1}} \frac{c_{2}}{|h|^{n+s-2}\left(1+\left|x^{\prime}\right|\right)^{n+k-s}} \mathrm{~d} x^{\prime} \mathrm{d} h<\infty
\end{align*}
$$

The conclusion follows by adding together inequalities (3.5) and (3.7).
Remark 3.2. Our approach to the trace inequalities given in Proposition 3.1 can be used to give a very simple argument to the classical Theorem 2.1. On the one hand, the variant of the Sobolev integral formula in Proposition 2.7 gives a representation formula in $\dot{\mathrm{W}}^{k, 1}$ supported on a pointed cone; on the other hand, the elementary argument in the proof of Proposition 3.1 reduces the traces inequalities to checking that $\mathrm{C}_{c}^{\infty}$ is contained in $\mathrm{L}^{1}$ and $\dot{\mathrm{B}}_{1,1}^{k-1}$.

## 4. Proof of the trace theorems

In this section, we prove the trace Theorems 1.2 and 1.3 , which we present here in unified form. Recall the space

$$
\mathrm{T}_{k}\left(H_{\nu}, V\right):=\left\{\left(f_{0}, f_{1}, \ldots, f_{k-1}\right): \begin{array}{c}
f_{k-1} \in \mathrm{~L}^{1}\left(H_{\nu}, V\right) \\
f_{j} \in \dot{\mathrm{~B}}_{1,1}^{k-1-j}\left(H_{\nu}, V\right), 0 \leqslant j \leqslant k-2
\end{array}\right\}
$$

which is Banach with respect to the canonical norm

$$
\left\|\left(f_{0}, f_{1}, \ldots, f_{k-1}\right)\right\|_{\mathrm{T}_{k}\left(H_{\nu}\right)}:=\left(\sum_{j=0}^{k-2}\left\|f_{j}\right\|_{\dot{\mathrm{B}}_{1,1}^{k-1-j}}\right)+\left\|f_{k-1}\right\|_{\mathrm{L}^{1}\left(H_{\nu}\right)}
$$

In the sequel, we denote for $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$

$$
\operatorname{tr}_{k} u:=\left.\left(u, \partial_{\nu} u, \ldots, \partial_{\nu}^{k-1} u\right)\right|_{H_{\nu}}
$$

We will now prove the following:
Theorem 4.1. Let $n \geq 2$ and $\mathbb{A}$ be a differential operator as in (1.1) of order $k \geq 1$. Then $\mathbb{A}$ is boundary elliptic in direction $\nu$ if and only if there exists a constant $c>0$ such that the estimate

$$
\left\|\operatorname{tr}_{k} u\right\|_{\mathrm{T}_{k}\left(H_{\nu}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)}
$$

holds for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
Remark 4.2. We remark that since $u$ admits traces in $\dot{\mathrm{B}}_{1,1}^{k-1}\left(H_{\nu}\right)$, all the derivatives of $u$ in the tangential direction have the suitable Besov regularity. Writing $D_{\tau}$ for the gradient in the tangential direction (of $H_{\nu}$ ), we have that $D_{\tau}^{j} u$ admits traces in $\dot{\mathrm{B}}_{1,1}^{k-1-j}\left(H_{\nu}\right)$ for all $j=0,1, \ldots, k-1$. We can thus write down as a corollary of Theorem 4.1 the following estimates for $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$ :

$$
\begin{aligned}
\left\|D^{j} u\right\|_{\dot{\mathrm{B}}_{1,1}^{k-j-1}\left(H_{\nu}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \quad \text { for } j=0,1, \ldots, k-2 \\
\left\|D_{\tau}^{k-1} u\right\|_{\dot{\mathrm{B}}_{1,1}^{0}\left(H_{\nu}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \\
\left\|\partial_{n}^{k-1} u\right\|_{\mathrm{L}^{1}\left(H_{\nu}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)}
\end{aligned}
$$

In particular, the only derivative of order at most $(k-1)$ for which the trace lacks Besov regularity is the pure $(k-1)$ th normal derivative.

For the remainder of the paper, we suppress the subscript from the notation for $H_{\nu}, H_{\nu}^{ \pm}$and write $x=\left(x^{\prime}, t\right)$ for a representation of $x \in \mathbb{R}^{n}$ in $H, H^{\perp}$ coordinates. We begin by proving necessity of boundary ellipticity.

Lemma 4.3. Let $n \geq 2$ and $\mathbb{A}$ be a differential operator as in (1.1) of order $k \geq 1$. Suppose that there exists a constant $c>0$ such that the estimate

$$
\begin{equation*}
\|u\|_{\dot{\mathrm{W}}^{k-1,1}\left(H_{\nu}\right)} \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \tag{4.1}
\end{equation*}
$$

holds for all $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$. Then $\mathbb{A}$ is boundary elliptic in direction $\nu$.
Proof. The proof relies on the fact that, if $\mathbb{A}$ is elliptic but not boundary elliptic in direction $\nu$, there exists $\eta \in \mathbb{R}^{n}$ such that $\mathbb{A}(\eta+\mathrm{i} \nu) v=0$ for some $v \in V+\mathrm{i} V$. We consider separately the cases when $\eta$ and $\nu$ are linearly independent or not.

If $\eta$ and $\nu$ are linearly independent, we will use coordinates $x=\left(x_{1}, x_{2}, x^{\prime \prime}\right) \in \mathbb{R}^{n}$, where $x_{1}=x \cdot \nu, x_{2}=x \cdot \eta$, and $x^{\prime \prime} \in\{\nu, \eta\}^{\perp}$. In this notation, we have that maps $u(x)=f\left(x_{1}+\mathrm{i} x_{2}\right) v$ satisfy $\mathbb{A} u(x)=0$ whenever $f$ is holomorphic at $x_{1}+\mathrm{i} x_{2}$ (see, e.g., [23, Lem. 3.2] or [9, Lem. 2.5]).

We will use an idea originating in the necessity proof of [4, Thm. V]. We choose $f=f_{\varepsilon}: \mathbb{C} \backslash(-\infty,-2 \varepsilon] \rightarrow \mathbb{C}$ be a primitive of $(z+2 \varepsilon)^{-1}$ for some $\varepsilon \in(0,1)$. We mean this in the following sense: let $f_{\varepsilon}^{(k-1)}(z)=(z+2 \varepsilon)^{-1}$, where the exponent denotes
$k-1$ complex derivatives. For $k>1$, this procedure requires choosing a branch of the logarithm, hence the restriction on the domain of $f_{\varepsilon}$.

We write $u_{\varepsilon}(x)=f_{\varepsilon}\left(x_{1}+\mathrm{i} x_{2}\right) v$. We consider cubes $Q_{\varepsilon}=(-\varepsilon, 1) \times(-1,1)^{n-1}$ and a cut-off function $\rho \in \mathrm{C}_{c}^{\infty}\left((-2,2)^{n}\right)$ such that $\rho=1$ in $[-1,1]^{n}$. We also choose $\varphi_{\varepsilon} \in \mathrm{C}^{\infty}\left(\mathbb{R}^{n}\right)$ be such that $\operatorname{spt} \varphi_{\varepsilon} \subset\left\{x \in \mathbb{R}^{n}: x_{1}>-2 \varepsilon\right\}$ and $\varphi_{\varepsilon}=1$ in $\left\{x \in \mathbb{R}^{n}: x_{1} \geq-\varepsilon\right\}$. Finally, we set $\psi_{\varepsilon}=\rho \varphi_{\varepsilon}$, which has the crucial properties that:
(i) $\psi_{\varepsilon}=1$ in $Q_{\varepsilon}$;
(ii) for $|\alpha| \leqslant k$, we have $\left|\partial^{\alpha} \psi_{\varepsilon}(x)\right|=\left\|\partial^{\alpha} \rho\right\|_{L^{\infty}}=c$ for $x \in H^{+}$;
(iii) $\psi_{\varepsilon} u_{\varepsilon} \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.

The latter implies that $\psi_{\varepsilon} u_{\varepsilon}$ is admissible for the estimate in (4.1). We compute:

$$
\mathbb{A}\left(\psi_{\varepsilon} u_{\varepsilon}\right)=\psi_{\varepsilon} \mathbb{A} u_{\varepsilon}+\sum_{\substack{|\alpha|+|\beta|=k \\|\alpha|<k}} c_{\alpha, \beta} \partial^{\alpha} u_{\varepsilon} \partial^{\beta} \psi_{\varepsilon}
$$

Since $\mathbb{A} u_{\varepsilon}=0$, this implies that

$$
\left\|\mathbb{A}\left(\psi_{\varepsilon} u_{\varepsilon}\right)\right\|_{\mathrm{L}^{1}\left(H^{+}\right)} \leqslant c \sum_{|\alpha| \leqslant k-1}\left\|\partial^{\alpha} u_{\varepsilon}\right\|_{\mathrm{L}^{1}\left(H^{+} \cap(-2,2)\right)^{n}} .
$$

Due to the structure of $f_{\varepsilon}$, the most singular term on the right hand side is no worse than

$$
\int_{(-2,2)^{2}} \frac{\mathrm{~d} x}{\left|\left(x_{1}, x_{2}\right)\right|}
$$

which is clearly finite. On the other hand, we have that

$$
\int_{H}\left|D^{k-1} u_{\varepsilon}\left(0, x_{2}, x^{\prime \prime}\right)\right| \mathrm{d}\left(x_{2}, x^{\prime \prime}\right) \geq \int_{[-1,1]^{n-2}} \int_{-1}^{1} \frac{\mathrm{~d} x_{2}}{\left|\left(2 \varepsilon, x_{2}\right)\right|} \mathrm{d} x^{\prime \prime} \sim \operatorname{arsinh}\left(\frac{1}{\varepsilon}\right) \rightarrow \infty
$$

as $\varepsilon \searrow 0$. Thus, we have obtained a contradiction when $\eta$ is not parallel to $\nu$.
If $\eta$ and $\nu$ are linearly dependent, then $\mathbb{A}(\nu) v=0$, and we proceed similarly to the previous case, defining now $u_{\varepsilon}(x)=g(x \cdot \nu) v$, with a function $g \in \mathrm{C}^{\infty}(\mathbb{R})$ chosen in such a way that $g(0)=1$.

The proof of the trace theorem is easily ensembled from the blocks we have:
Proof of Theorem 4.1. The necessity of boundary ellipticity follows at once from Lemma 4.3. Assume next that $\mathbb{A}$ is boundary elliptic in direction $\nu$. We can identify $H_{\nu}$ with $\mathbb{R}^{n-1}$ and $H_{\nu}^{+}$with $\mathbb{R}_{+}^{n}$. Let $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$, so by Theorem 2.6 , we have that we can write

$$
u(x)=\int_{\mathbb{R}_{+}^{n}} K(x+y) \mathbb{A} u(y) \mathrm{d} y
$$

where $K$ is smooth away from zero, $(k-n)$-homogeneous and vanishes on $\mathbb{R}_{-}^{n}$. Let $j=0,1, \ldots, k-1$. Then

$$
\partial_{n}^{j} u(x)=\int_{\mathbb{R}_{+}^{n}} K_{j}(x+y) \mathbb{A} u(y) \mathrm{d} y, \quad \text { where } K_{j}=\partial_{n}^{j} K
$$

Therefore $K_{j}$ is smooth away from zero, $(k-j-n)$-homogeneous and vanishes on $\mathbb{R}_{-}^{n}$. We can thus apply Proposition 3.1 with $s=k-j \geq 1$ and obtain the trace inequalities

$$
\begin{aligned}
\left\|\partial_{n}^{j} u\right\|_{\dot{\mathrm{B}}_{1,1}^{k-j-1}\left(\mathbb{R}^{n-1}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)} \quad \text { for } j=0,1, \ldots, k-2 \\
\left\|\partial_{n}^{k-1} u\right\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n-1}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(\mathbb{R}_{+}^{n}\right)}
\end{aligned}
$$

The proof is complete.

## 5. Proof of the Sobolev estimate

In this section we prove the main Theorem 1.1. The proof follows from the trace Theorems 1.2 and 1.3 and the fact that boundary ellipticity implies cancellation. This latter observation is presented in the following:

Proposition 5.1. If $n \geq 2$, if $\mathbb{A}$ be an operator which is elliptic and boundary elliptic in some direction $\nu \in \mathbb{S}^{n-1}$, then $\mathbb{A}$ is canceling, i.e., $\mathbb{A}$ satisfies (C).

The assumption that $n \geq 2$ is essential, as for $n=1$ there are no canceling operators.
Proof of Proposition 5.1. Without loss of generality, we can say that $\mathbb{A}$ is boundary elliptic in direction $e_{1}$, where $\left\{e_{j}\right\}_{j=1}^{n}$ is a standard orthonormal basis of $\mathbb{R}^{n}$. Define the operator $\mathbb{A}_{1}$ by $\mathbb{A}_{1}(\xi)=\mathbb{A}(\xi)$ for $\xi \in \operatorname{span}\left\{e_{1}, e_{2}\right\} \simeq \mathbb{R}^{2}$. By definition, $\mathbb{A}_{1}$ is elliptic and boundary elliptic in direction $e_{1}$ on $\mathbb{R}^{2}$. We claim that $\mathbb{A}_{1}$ is $\mathbb{C}$-elliptic, i.e.,

$$
\begin{equation*}
\operatorname{ker}_{\mathbb{C}} \mathbb{A}_{1}\left(\xi_{1}+\mathrm{i} \xi_{2}\right)=\{0\} \tag{5.1}
\end{equation*}
$$

for all linearly independent $\xi_{1}, \xi_{2} \in \mathbb{R}^{2}$ (if $\xi_{1}, \xi_{2}$ are not linearly independent, then (5.1) follows by ellipticity and homogeneity of $\mathbb{A}$ ). By linear independence of $\xi_{i}$, we can find $\lambda \in \mathbb{C} \backslash\{0\}$ such that $\lambda\left(\xi_{1}+\mathrm{i} \xi_{2}\right)=\xi+\mathrm{i} e_{1}$, for some $\xi \in \mathbb{R}^{2}$. In fact, writing $\xi_{1}=\left(\xi_{11}, \xi_{21}\right)$ and $\xi_{2}=\left(\xi_{12}, \xi_{22}\right)$, we have $\lambda\left(\xi_{1}+\mathrm{i} \xi_{2}\right)=\xi+\mathrm{i} e_{1}$ for some $\lambda=\operatorname{Re}(\lambda)+\mathrm{i} \operatorname{Im}(\lambda) \in \mathbb{C} \backslash\{0\}$ and some $\xi \in \mathbb{R}^{2}$ if and only if

$$
\left\{\begin{array}{l}
\xi_{11} \operatorname{Im}(\lambda)+\xi_{12} \operatorname{Re}(\lambda)=1 \\
\xi_{21} \operatorname{Im}(\lambda)+\xi_{22} \operatorname{Re}(\lambda)=0
\end{array}\right.
$$

and this is clearly solvable for $(\operatorname{Re}(\lambda), \operatorname{Im}(\lambda)) \in \mathbb{R}^{2} \backslash\{0\}$ by the linear independence of $\xi_{1}, \xi_{2}$. Now, by homogeneity of $\mathbb{A}$, we have that $\operatorname{ker}_{\mathbb{C}} \mathbb{A}_{1}\left(\xi_{1}+\mathrm{i} \xi_{2}\right)=\operatorname{ker}_{\mathbb{C}} \mathbb{A}_{1}\left(\xi+\mathrm{i} e_{1}\right)$, so that (5.1) holds by boundary ellipticity of $\mathbb{A}_{1}$ in direction $e_{1}$.

It then follows by [23, Lem. 3.2] or [24] that $\mathbb{A}_{1}$ is canceling, so that

$$
\bigcap_{\xi \in \mathbb{R}^{n} \backslash\{0\}} \operatorname{im} \mathbb{A}(\xi) \subset \bigcap_{\xi \in \operatorname{span}\left\{e_{1}, e_{2}\right\} \backslash\{0\}} \operatorname{im} \mathbb{A}(\xi)=\bigcap_{\xi \in \operatorname{span}\left\{e_{1}, e_{2}\right\} \backslash\{0\}} \operatorname{im} \mathbb{A}_{1}(\xi)=\{0\}
$$

which concludes the proof.
We can now proceed with the
Proof of Theorem 1.1. We will now show that (b) holds, using both (a) and Theorem 4.1. Let $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$. Using the extension Theorem 2.2 , we find $U \in \dot{\mathrm{~W}}^{k, 1}\left(H^{-}, V\right)$ such that $\left.D^{j} U\right|_{H}=\left.D^{j} u\right|_{H}$ for $j=0,1, \ldots k-1$ (cf. Remark 4.2) and

$$
\begin{equation*}
\left\|D^{k} U\right\|_{\mathrm{L}^{1}\left(H^{-}\right)} \leqslant c\left\|\operatorname{tr}_{k} u\right\|_{\mathrm{T}_{k}(H)} \tag{5.2}
\end{equation*}
$$

Define an extension operator $E u$ by $u$ in $H^{+}$and by $U$ in $H^{-}$. We now check that $\mathbb{A} E u \in \mathrm{~L}^{1}\left(\mathbb{R}^{n}, W\right)$. This will enable us to use the full-space estimate [53, Thm. 1.3]. Let $\varphi \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, W\right)$. Then, letting $\mathbb{A}^{*}:=(-1)^{k} \sum_{|\alpha|=k} A_{\alpha}^{*} \partial^{\alpha}$ be the formal adjoint of $\mathbb{A}$, we conclude

$$
\begin{align*}
\int_{\mathbb{R}^{n}} E u \cdot \mathbb{A}^{*} \varphi \mathrm{~d} x & =\int_{H^{+}} u \cdot \mathbb{A}^{*} \varphi \mathrm{~d} x+\int_{H^{-}} U \cdot \mathbb{A}^{*} \varphi \mathrm{~d} x \\
& =\int_{H^{+}} \mathbb{A} u \cdot \varphi \mathrm{~d} x+\int_{H^{-}} \mathbb{A} U \cdot \varphi \mathrm{~d} x \tag{5.3}
\end{align*}
$$

where the boundary terms in the integration by parts vanish since the traces of $D^{j} u$ and $D^{j} U$ coincide for $j=0,1, \ldots, k-1$ by construction. As a consequence, $\mathbb{A} E u \in \mathrm{~L}^{1}\left(\mathbb{R}^{n}, W\right)$ with

$$
\mathbb{A} E u= \begin{cases}\mathbb{A} u & \text { in } H^{+} \\ \mathbb{A} U & \text { in } H^{-}\end{cases}
$$

We then estimate from (5.2) and (5.3)

$$
\begin{aligned}
\left\|D^{k-1} u\right\|_{\mathrm{L}^{\frac{n}{n-1}}\left(H^{+}\right)} & \leqslant\left\|D^{k-1} E u\right\|_{\mathrm{L}^{\frac{n}{n-1}}\left(\mathbb{R}^{n}\right)} \leqslant c\|\mathbb{A} E u\|_{\mathrm{L}^{1}\left(\mathbb{R}^{n}\right)} \\
& =c\left(\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H^{+}\right)}+\|\mathbb{A} U\|_{\mathrm{L}^{1}\left(H^{-}\right)}\right) \\
& \leqslant c\left(\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H^{+}\right)}+\left\|\operatorname{tr}_{k} u\right\|_{\mathrm{T}_{k}\left(H^{-}\right)}\right) \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H^{+}\right)}
\end{aligned}
$$

where in the second inequality we used [53, Thm. 1.3] and the fact that boundary ellipticity implies cancellation (see Proposition 5.1); in the last inequality we used Theorem 4.1. The proof of (b) is complete.

To prove the converse, first note that necessity of ellipticity for the estimate follows from [53, Cor. 5.2]. Therefore, assume that $\mathbb{A}$ is elliptic, but not boundary elliptic. We conclude the proof of the main result by showing that the estimate in (b) must fail. We keep most of the construction in the proof of Lemma 4.3, with the only modification that $f_{\varepsilon}: \mathbb{C} \backslash(-\infty,-2 \varepsilon] \rightarrow \mathbb{C}$ is given by $f_{\varepsilon}^{(k-1)}(z)=(z+2 \varepsilon)^{\alpha}, \alpha=-\frac{2(n-1)}{n}$, where we choose a branch of $z^{-\alpha}$ according to the domain of $f_{\varepsilon}$. The remaining details are left to the keen reader.

In following the same ideas and using the full space estimates for (weakly) canceling operators in $[53,8,40]$, we can prove a broader class of estimates:

Theorem 5.2. Let $n \geq 2$ and $\mathbb{A}$ be a kth order differential operator as in (1.1). Suppose that $\mathbb{A}$ is elliptic and boundary elliptic in direction $\nu$. Let $s \in(0, n)$ be such that $s \leqslant k, j=1,2, \ldots, \min \{k, n-1\}$, and $q \in[1, n /(n-j)]$. Then the following estimates hold

$$
\begin{aligned}
\|u\|_{\dot{\mathrm{W}}^{k-s, \frac{n}{n-s}\left(H_{\nu}^{+}\right)}} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \\
\left\||\cdot|^{n-j-n / q} D^{k-j} u\right\|_{\mathrm{L}^{q}\left(H_{\nu}^{+}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \\
\left\|D^{k-n} u\right\|_{\mathrm{L}^{\infty}\left(H_{\nu}^{+}\right)} & \leqslant c\|\mathbb{A} u\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \quad \text { when } k \geq n
\end{aligned}
$$

for $u \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n}, V\right)$.
Here we make the convention that the fractional scale $\dot{W}^{s, p}$ is completed with the classical Sobolev spaces when $s$ is a positive integer and with the Lebesgue spaces when $s=0$.

Remark 5.3. The $\mathrm{L}^{\infty}$-estimate in Theorem 5.2 can be proved independently of the machinery used to prove Theorem 1.1. One can simply use the representation formula of Theorem 2.6 to note that $D^{k-n} u$ can be represented by the convolution of a bounded kernel with $\mathbb{A} u$. In particular, the $L^{\infty}$-estimate is true in the absence of ellipticity, which is necessary for the higher order estimate of Theorem 1.1.

## 6. Boundary ellipticity, trace operators and examples

In this section we classify the boundary ellipticity among related conditions that lead to trace or Sobolev-type inequalities on domains or the entire space, respectively, display the consequences for spaces of functions defined in terms of the differential operators $\mathbb{A}$ and discuss several examples.

Based on Proposition 5.1, we first obtain the following implications for $\nu \in \mathbb{S}^{n-1}$ and operators $\mathbb{A}$ of the form (1.1):
(6.1) $\mathbb{A}$ is $\mathbb{C}$-elliptic $\Longrightarrow \mathbb{A}$ is boundary elliptic in direction $\nu \Longrightarrow \mathbb{A}$ is canceling.

To connect the consequences of the preceding chain of implications with previously known results, we first restate the inequalities of the preceding sections in the language of trace operators. From a function space perspective and in order to provide a unifying
framework for problems arising, e.g., in elasticity or plasticity [21, 47], it is convenient to put for an open set $\Omega \subset \mathbb{R}^{n}$

$$
\begin{align*}
& \mathrm{W}^{\mathbb{A}, 1}(\Omega):=\left\{u \in \mathrm{~W}^{k-1,1}(\Omega, V): \mathbb{A} u \in \mathrm{~L}^{1}(\Omega, W)\right\} \\
& \operatorname{BV}^{\mathbb{A}}(\Omega):=\left\{u \in \mathrm{~W}^{k-1,1}(\Omega, V): \mathbb{A} u \in \mathcal{M}(\Omega, W)\right\} \tag{6.2}
\end{align*}
$$

where $\mathcal{M}(\Omega, W)$ denotes the $W$-valued Radon measures $\mu$ with finite total variation $\|\mu\|_{\mathcal{M}(\Omega)}$ on $\Omega$. We define the corresponding norms or metrics on $\mathrm{W}^{\mathbb{A}, 1}$ or $\mathrm{BV}^{\mathbb{A}}$, respectively, by

$$
\begin{array}{lr}
\|u\|_{\mathrm{W}^{\mathbb{A}, 1}(\Omega)}:=\|u\|_{\mathrm{W}^{k-1,1}(\Omega)}+\|\mathbb{A} u\|_{\mathrm{L}^{1}(\Omega)} & \text { for } u \in \mathrm{~W}^{\mathbb{A}, 1}(\Omega), \\
\|u\|_{\mathrm{BV}^{\mathbb{A}}(\Omega)}:=\|u\|_{\mathrm{W}^{k-1,1}(\Omega)}+\|\mathbb{A} u\|_{\mathcal{M}(\Omega)} & \text { for } u \in \operatorname{BV}^{\mathbb{A}}(\Omega), \\
d_{\mathbb{A}}(u, v):=\|u-v\|_{\mathrm{W}^{k-1,1}(\Omega)}+\left|\|\mathbb{A} u\|_{\mathcal{M}(\Omega)}-\|\mathbb{A} v\|_{\mathcal{M}(\Omega)}\right| & \text { for } u, v \in \operatorname{BV}^{\mathbb{A}}(\Omega),
\end{array}
$$

and note that approximation by smooth functions of $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ can only be expected with respect to $d_{\mathbb{A}}$ but not $\|\cdot\|_{B^{\mathbb{A}}}$ (see e.g., [9, Sec. 2.4] and [41, Sec. 2.3]). Because of this circumstance, we give the detailled proof of

Corollary 6.1 (Refined trace theorem). Let $\mathbb{A}$ as in (1.1) be a kth order operator that is boundary elliptic in direction $\nu \in \mathbb{S}^{n-1}$. Then there exists a surjective, linear trace operator $\operatorname{tr}_{k}: \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right) \rightarrow \mathrm{T}_{k}\left(H_{\nu}, V\right)$ which is continuous with respect to $d_{\mathbb{A}}$. More precisely, there exists $c=c(\mathbb{A}, \nu)>0$ such that the estimate

$$
\begin{equation*}
\left\|\operatorname{tr}_{k}(u)\right\|_{\mathrm{T}_{k}\left(H_{\nu}\right)} \leqslant c\|\mathbb{A} u\|_{\mathcal{M}\left(H_{\nu}^{+}\right)} \tag{6.3}
\end{equation*}
$$

holds for all $u \in \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)$.
Proof. Given $u \in \mathrm{~W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$, we may follow [18, §5.3.3] and consider for $\varepsilon>0$ the maps $u^{\varepsilon}(x):=u(x+\varepsilon \nu)$ for $x \in H_{\nu}^{+}$. Passing to the mollifications $u_{\varepsilon}:=\rho_{\varepsilon / 2} * u^{\varepsilon}$ with the $\varepsilon$-rescaled variant of a standard mollifier $\rho$ then yields that $u_{\varepsilon} \in \mathrm{C}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right)$ and $\left\|u_{\varepsilon}-u\right\|_{\mathrm{W}^{\mathrm{A}, 1}\left(H_{\nu}^{+}\right)} \rightarrow 0$ as $\varepsilon \searrow 0$. Hence $\mathrm{C}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right) \cap \mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$is dense in $\mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$ for the norm topology. On the other hand, whenever $\eta_{R} \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{n} ;[0,1]\right)$ satisfies $\mathbb{1}_{B_{R}(0)} \leqslant \eta_{R} \leqslant \mathbb{1}_{B_{2 R}(0)}$ together with

$$
\left|\nabla^{l} \eta_{R}\right| \leqslant \frac{c}{R^{l}} \quad \text { for all } l \in\{0, \ldots, k\}
$$

then $\eta_{R} u \rightarrow u$ as $R \rightarrow \infty$ for the norm topology on $\mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$. Combining both statements yields that $\mathrm{C}_{c}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right)$ is dense in $\mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$for the norm topology.

For $u \in \mathrm{~W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$, we may thus pick a sequence $\left(u_{j}\right) \subset \mathrm{C}_{c}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right)$ such that $u_{j} \rightarrow u$ for the $\mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$-norm. Theorem 4.1 then implies that $\left(\operatorname{tr}_{k} u_{j}\right)$ is Cauchy in $\mathrm{T}_{k}\left(H_{\nu}, V\right)$ and, for $\mathrm{T}_{k}\left(H_{\nu}, V\right)$ is Banach, converges to some element $\operatorname{tr}_{k} u \in \mathrm{~T}_{k}\left(H_{\nu}, V\right)$. By a routine argument, one sees that this element $\operatorname{tr}_{k} u$ is independent of the approximating sequence and thus well-defined. This defines a linear and bounded trace operator $\operatorname{tr}_{k}: \mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right) \rightarrow \mathrm{T}_{k}\left(H_{\nu}, V\right)$, and this operator satisfies (6.3) in light of Theorem 4.1.

For $u \in \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)$, we choose a sequence $\left(v_{j}\right) \subset \mathrm{C}^{\infty}\left(H_{\nu}^{+}, V\right) \cap \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right) \subset$ $\mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$such that $d_{\mathbb{A}}\left(v_{j}, u\right) \rightarrow 0$ as $j \rightarrow \infty$, see [9, Sec. 2.4] or [41, Sec. 2.3]. Let $r>0$ and pick a cut-off function $\varphi_{r} \in \mathrm{C}^{\infty}\left(H_{\nu}^{+} ;[0,1]\right)$ with $\varphi_{r}(x)=1$ for $x \in H_{\nu}^{+}$with $\operatorname{dist}\left(x, H_{\nu}\right)<r, \varphi_{r}(x)=0$ for $x \in H_{\nu}^{+}$with $x \in H_{\nu}^{+} \backslash S_{r}$, where $S_{r}:=\left\{x \in H_{\nu}^{+}: \operatorname{dist}\left(x, H_{\nu}\right) \leqslant 2 r\right\}$ and

$$
\begin{equation*}
\left|\nabla^{l} \varphi_{r}\right| \leqslant \frac{c}{r^{l}} \quad \text { for } l \in\{0, \ldots, k\} \tag{6.4}
\end{equation*}
$$

By the construction of $\operatorname{tr}_{k}: \mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right) \rightarrow \mathrm{T}_{k}\left(H_{\nu}, V\right)$, we have $\operatorname{tr}_{k}\left(v_{j}\right)=\operatorname{tr}_{k}\left(\varphi_{r} v_{j}\right)$ for all $j \in \mathbb{N}$ and all $r>0$. Using (6.3) for $\varphi_{r}\left(v_{i}-v_{k}\right) \in \mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$as established above
in the second step, we obtain for all $i, j \in \mathbb{N}$ by the Leibniz rule

$$
\begin{align*}
& \left\|\operatorname{tr}_{k}\left(v_{i}-v_{j}\right)\right\|_{\mathrm{T}_{k}\left(H_{\nu}\right)}=\left\|\operatorname{tr}_{k}\left(\varphi_{r}\left(v_{i}-v_{j}\right)\right)\right\|_{\mathrm{T}_{k}\left(H_{\nu}\right)} \\
& \quad \stackrel{(6.4)}{\leqslant} c\left\|\mathbb{A}\left(\varphi_{r}\left(v_{i}-v_{j}\right)\right)\right\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)} \\
& \quad \leqslant c\left(\sum_{m=0}^{k-1} \frac{1}{r^{k-m}}\left\|D^{m}\left(v_{i}-v_{j}\right)\right\|_{\mathrm{L}^{1}\left(H_{\nu}^{+}\right)}\right)+c\left(\left|\mathbb{A} v_{i}\right|\left(S_{r}\right)+\left|\mathbb{A} v_{j}\right|\left(S_{r}\right)\right) \tag{6.5}
\end{align*}
$$

First letting $i, j \rightarrow \infty$ and then sending $r \searrow 0$, we see that $\left(\operatorname{tr}_{k}\left(v_{j}\right)\right)$ is Cauchy in $\mathrm{T}_{k}\left(H_{\nu}, V\right)$ and thus converges to some element of $\mathrm{T}_{k}\left(H_{\nu}, V\right)$. By an argument similar to (6.5), one equally finds that this element is independent of the approximating sequence $\left(v_{j}\right)$, so is well-defined, and depends linearly on $u$; note that, even though $d_{\mathbb{A}}$ is not translation invariant, the linearity can be obtained by a similar argument as invoked in (6.5). This defines the requisite trace operator $\operatorname{tr}_{k}: \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right) \rightarrow \mathrm{T}_{k}\left(H_{\nu}, V\right)$ which, by construction is continuous for $d_{\mathbb{A}}$. By construction, it coincides with the trace operator $\mathrm{W}^{k, 1}\left(H_{\nu}^{+}, V\right) \rightarrow \mathrm{T}_{k}\left(H_{\nu}, V\right)$ on $\mathrm{W}^{k, 1}\left(H_{\nu}^{+}, V\right)$-maps, and hence its surjectivity follows from Theorem 2.2. The proof is complete.
Corollary 6.2 (Refined Sobolev-type inequalities). Let $\mathbb{A}$ as in (1.1) be a kth order elliptic operator that is boundary elliptic in direction $\nu$. Then there exists $c=c(\mathbb{A}, \nu)>$ 0 and, for any $0<s<1$, a constant $c_{s}=c(\mathbb{A}, \nu, s)>0$ such that the Sobolev-type estimates

$$
\begin{align*}
\left\|D^{k-1} u\right\|_{\mathrm{L}^{\frac{n}{n-1}}\left(H_{\nu}^{+}\right)} \leqslant c\|\mathbb{A} u\|_{\mathcal{M}\left(H_{\nu}^{+}\right)} \\
\left\|D^{k-1} u\right\|_{\dot{\mathrm{W}}^{s, \frac{n}{n-1+s}\left(H_{\nu}^{+}\right)}} \leqslant c_{s}\|\mathbb{A} u\|_{\mathcal{M}\left(H_{\nu}^{+}\right)} \tag{6.6}
\end{align*}
$$

hold for all $u \in \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)$.
Proof. Let $u \in \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)$and pick a sequence $\left(w_{j}\right) \subset \mathrm{C}_{c}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right)$ that converges to $u$ with respect to $d_{\mathbb{A}}$; this can be achieved by taking a sequence $\left(v_{j}\right) \subset \mathrm{W}^{\mathbb{A}, 1}\left(H_{\nu}^{+}\right)$such that $d_{\mathbb{A}}\left(v_{j}, u\right)<\frac{1}{j}$ and then choosing, for each $j \in \mathbb{N}$, some $w_{j} \in \mathrm{C}_{c}^{\infty}\left(\overline{H_{\nu}^{+}}, V\right)$ such that $\left\|w_{j}-v_{j}\right\|_{\mathrm{W}^{\mathrm{A}, 1}\left(H_{\nu}^{+}\right)}<\frac{1}{j}$ as in the very first part of the previous proof. Passing to a subsequence if necessary, we may achieve $D^{k-1} w_{j} \rightarrow D^{k-1} u \mathscr{L}^{n}$-a.e. in $H_{\nu}^{+}$; then (6.6) is a direct consequence of Fatou's lemma, Theorem 5.2 and $d_{\mathbb{A}}\left(w_{j}, u\right) \rightarrow 0$ as $j \rightarrow \infty$.

We now turn to some examples that demonstrate the richness and the limitations of the boundary ellipticity.

Example 6.3 (C-elliptic operators). Based on (6.1), all $\mathbb{C}$-elliptic operators are boundary elliptic in any direction $\nu \in \mathbb{S}^{n-1}$. This particularly comprises the symmetric gradient

$$
\begin{equation*}
\varepsilon(u):=\frac{1}{2}\left(D u+D u^{\top}\right), \quad u=\left(u_{1}, \ldots, u_{n}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \tag{6.7}
\end{equation*}
$$

for $n \geq 2$ and, denoting by $\mathbb{1}_{n \times n}$ the ( $n \times n$ )-unit matrix, the trace-free symmetric or deviatoric symmetric gradient

$$
\begin{equation*}
\varepsilon^{D}(u):=\varepsilon(u)-\frac{\operatorname{div}(u)}{n} \mathbb{1}_{n \times n}, \quad u=\left(u_{1}, \ldots, u_{n}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \tag{6.8}
\end{equation*}
$$

in $n \geq 3$ dimensions (see, e.g., [9, Sec. 2]). For the symmetric gradient, Corollary 6.1 directly yields the halfspace version of the BD-trace theorem due to Strang-Temam [47] (also see Babadjian [5]); more generally, for halfspaces Corollary 6.1 lets us retrieve the trace theorems for $\mathbb{C}$-elliptic operators from $[9,17,24]$ as special cases by (6.1).

Example 6.4 (The trace-free symmetric gradient in $n=2$ dimensions). If $n=2$, then $\mathbb{C}$-ellipticity coincides with cancellation for first order elliptic operators, see [23]. The trace-free symmetric gradient (6.8) is known to be non-canceling for $n=2$ and


Figure 2. Shifting holomorphic maps along $\mathbb{R}^{n-2}$. To construct domains for which there is no trace operator $\operatorname{BV}^{\mathbb{A}}(\Omega) \rightarrow \mathrm{T}_{k}(\partial \Omega, V)$ in absence of $\mathbb{C}$-ellipticity, one picks $\xi \in \mathbb{C}^{n} \backslash\{0\}$ and $v \in(V+\mathrm{i} V) \backslash\{0\}$ such that $\mathbb{A}(\xi) v=0$. For suitable holomorphic functions $f: \mathbb{C} \supset \mathbb{D} \rightarrow$ $\mathbb{C}$ (e.g. with $f^{(k-1)}(z)=\frac{1}{z-1}$ and the complex disk $\left.\mathbb{D}\right)$, either the real or the imaginary part of $u(x):=f(x \cdot \operatorname{Re}(\xi)+\mathrm{i} x \cdot \operatorname{Im}(\xi)) v$ violate the trace estimate over a set $\Omega$ that up to a rotation coincides with $\left\{t_{1} \operatorname{Re}(\xi)+t_{2} \operatorname{Im}(\xi)+\left(0, z^{\prime \prime}\right): t_{1}^{2}+t_{2}^{2}<1, z^{\prime \prime} \in \mathbb{R}^{n-2}\right\}$ (figure to the left); see [9, Thm. 4.18], [24]. In the same way, one can come up with domains that violate the the trace estimate for non-boundaryelliptic operators by use of Lemma 4.3 (figure to the right). Including a straight piece orthogonal to $\operatorname{Im}(\xi)$, one sees the necessity of the boundary ellipticity even more directly.
therefore, in light of (6.1), cannot be boundary elliptic in direction $\nu$ for any $\nu \in \mathbb{S}^{1}$. We may then explicitly verify that for no halfspace $H_{\nu}^{+} \subset \mathbb{R}^{2}$ the operator $\varepsilon^{D}$ admits the trace or Sobolev estimates from Corollary 6.1 and 6.2: Put

$$
f\left(x_{1}, x_{2}\right):=\left(\frac{x_{1}}{x_{1}^{2}+x_{2}^{2}},-\frac{x_{2}}{x_{1}^{2}+x_{2}^{2}}\right), \quad\left(x_{1}, x_{2}\right) \in H_{\nu}^{+}
$$

An explicit computation directly verifies that $\varepsilon^{D}(f)\left(x_{1}, x_{2}\right)=0 \in \mathbb{R}^{2 \times 2}$ for all $\left(x_{1}, x_{2}\right) \in H_{\nu}^{+}$regardless of $\nu=\left(\nu_{1}, \nu_{2}\right) \in \mathbb{S}^{1}$. Whenever $\eta_{R} \in \mathrm{C}_{c}^{\infty}\left(\mathbb{R}^{2} ;[0,1]\right)$ is a cut-off function with $\mathbb{1}_{B_{R}(0)} \leqslant \eta_{R} \leqslant \mathbb{1}_{B_{2 R}(0)}$ and $\left|\nabla \eta_{R}\right| \leqslant \frac{2}{R}$ for $R>0$, then $u_{R}:=$ $\eta_{R} f \in \mathrm{BV}^{\mathbb{A}}\left(H_{\nu}^{+}\right)$. By construction, we obtain that $\sup _{R>0}\left\|\varepsilon^{D}\left(u_{R}\right)\right\|_{\mathcal{M}\left(H_{\nu}^{+}\right)}<\infty$, but parametrising $H_{\nu}=\mathbb{R} \nu^{\perp}$ with $\nu^{\perp}=\left(-\nu_{2}, \nu_{1}\right)$, we then obtain

$$
\int_{H_{\nu}}\left|u_{R}\left(x_{1}, x_{2}\right)\right| \mathrm{d} \mathscr{H}^{1}\left(x_{1}, x_{2}\right) \geq \int_{-R}^{R} \frac{\mathrm{~d} t}{|t|}=\infty
$$

which is in line with Corollary 6.1. Similarly, one obtains with a constant $c>0$

$$
\int_{H_{\nu}^{+}}\left|u_{R}(x)\right|^{2} \mathrm{~d} x \geq \int_{H_{\nu}^{+} \cap B_{R}(0)} \frac{\mathrm{d}\left(x_{1}, x_{2}\right)}{x_{1}^{2}+x_{2}^{2}} \geq c \int_{0}^{R} \frac{\mathrm{~d} r}{r}=\infty
$$

which is in line with Corollary 6.2.
Even though our main focus of the present paper is on halfspaces, let us note that the failure of boundary ellipticity of $\mathbb{A}$ in a certain direction can immediately be used to construct a domain $\Omega \subset \mathbb{R}^{n}$ for which there is no boundary trace operator $\mathrm{BV}^{\mathbb{A}}(\Omega) \rightarrow \mathrm{T}_{k}(\partial \Omega, V)$ (see Figure 2) with the obvious definition of the latter space via local charts. However, as it is more restrictive for an operator to not be boundary elliptic in a certain direction than to not be $\mathbb{C}$-elliptic in general (see the next example), a modification of the argument sketched in Figure 2 directly yields that the existence of a trace operator $\mathrm{BV}^{\mathbb{A}}(\Omega) \rightarrow \mathrm{T}_{k}(\partial \Omega, V)$ forces the outward unit normals $\nu_{\partial \Omega}$ to belong
to some set $K \subset \mathbb{S}^{n-1}$ depending on $\mathbb{A}$. While this technical point will be pursued in future work, we conclude the present paper by giving examples of operators that, in view of (6.1), fail to be $\mathbb{C}$-elliptic, yet are boundary elliptic in certain directions and thus admit Sobolev estimates on the corresponding halfspaces:

Example 6.5. Let $n \geq 3, N \geq 3, V=\mathbb{R}^{N}, W=\mathbb{R}^{((N-1) n-1) \times 2}$ and consider the differential operator $\mathbb{A}$ acting on $u=\left(u_{1}, \ldots, u_{N}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{N}$
$\mathbb{A} u=\left[\begin{array}{c|c}\partial_{1} u_{1}-\partial_{2} u_{2} & \partial_{1} u_{2}+\partial_{2} u_{1} \\ \partial_{3} u_{1} & \partial_{3} u_{2} \\ \vdots & \vdots \\ \partial_{n} u_{1} & \partial_{n} u_{2} \\ \hline \nabla u_{3} & \mathbf{0} \\ \vdots & \vdots \\ \nabla u_{N} & \mathbf{0}\end{array}\right]$,
where $\mathbf{0}$ denotes the zero vector in $\mathbb{R}^{n}$. As established in [23, Counterexample 3.4], this operator serves as an example of an elliptic operator being canceling yet failing to be $\mathbb{C}$-elliptic. However, $\mathbb{A}$ is boundary elliptic in every direction $\nu \in \operatorname{span}\left\{e_{3}, \ldots, e_{n}\right\}$. Based on this operator, boundary elliptic, non- $\mathbb{C}$-elliptic operators of arbitrary order can be constructed: In fact, if $\mathbb{B}$ is a $(k-1)$ th order, $\mathbb{C}$-elliptic differential operator on $\mathbb{R}^{n}$ from $W=\mathbb{R}^{((N-1) n-1) \times 2}$ to some finite dimensional real vector space, then $\mathbb{B} \mathbb{A}$ is of $k$ th order, boundary elliptic in all directions $\nu \in \operatorname{span}\left\{e_{3}, \ldots, e_{n}\right\}$ but non- $\mathbb{C}$-elliptic.

## References

[1] Adams, R. A., 1975. Sobolev spaces. Pure and Applied Mathematics, Vol. 65. Academic Press, New York-London.
[2] Agmon, S., Douglis, A., and Nirenberg, L., 1959. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. Communications on pure and applied mathematics, 12(4), pp.623-727.
[3] Agmon, S., Douglis, A., and Nirenberg, L., 1964. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Communications on pure and applied mathematics, 17(1), pp.35-92.
[4] Aronszajn, N., 1954. On coercive integro-differential quadratic forms. In Conference on partial differential equations, University of Kansas (pp. 94-106).
[5] Babadjian, J.-F., 2015. Traces of functions of bounded deformation. Indiana Univ. Math. J. 64, no. 4, 1271-1290.
[6] Bourgain, J. and Brezis, H., 2003. On the equation div $Y=f$ and application to control of phases. Journal of the American Mathematical Society, 16(2), pp.393-426.
[7] Bourgain, J. and Brezis, H., 2007. New estimates for elliptic equations and Hodge type systems. Journal of the European Mathematical Society, 9(2), pp.277-315.
[8] Bousquet, P. and Van Schaftingen, J., 2014. Hardy-Sobolev inequalities for vector fields and canceling linear differential operators. Indiana Univ. Math. J. 63 (5), pp.1419-1445.
[9] Breit, D., Diening, L., and Gmeineder, F., 2020. On the trace operator for functions of bounded $\mathbb{A}$ - variation. Analysis \& PDE, Vol. 13 (2020), No. 2, 559-594.
[10] Brezis, H. and Van Schaftingen, J., 2007. Boundary estimates for elliptic systems with $\mathrm{L}^{1}$-data. Calculus of Variations and Partial Differential Equations, 30(3), pp.369-388.
[11] Calderón, A.P. and Zygmund, A., 1952. On the existence of certain singular integrals. Acta Mathematica 88, pp. 85-139.
[12] Calderón, A.P. and Zygmund, A., 1956. On singular integrals. Amer. J. Math., 78:289-309.
[13] Cox, D., Little, J. and O'Shea, D., 2015. Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra. Fourth edition. Undergraduate Texts in Mathematics. Springer, Cham.
[14] Denk, R., Hieber, M. and Prüss, J.: Towards an $\mathrm{L}^{1}$-theory for vector-valued elliptic boundary value problems. Progr. Nonlinear Differential Equations Appl. 55 (2003), 141-147.
[15] De Leeuw, K. and Mirkil, H., 1964. A priori estimates for differential operators in $\mathrm{L}_{\infty}$ norm. Illinois Journal of Mathematics, 8(1), pp.112-124.
[16] Diening, L. and Gmeineder, F., 2020. Continuity points via Riesz potentials for $\mathbb{C}$-elliptic operators. Quarterly Journal of Mathematics, 71(4):1201-1218.
[17] Diening, L. and Gmeineder, F., 2021. Sharp trace and Korn inequalities for differential operators. ArXiv preprint, ArXiv 2105.09570.
[18] Evans, L.C., 1998. Partial differential equations. Graduate Studies in Mathematics 19, American Mathematical Society.
[19] Faraco, D. and Guerra, A., 2021. Remarks on Ornstein's Non-Inequality in $\mathbb{R}^{2 \times 2}$. Quart. J. Math. 00, 1-5.
[20] Friedrichs, K.O, 1947. On the boundary-value problems of the theory of elasticity and Korn's inequality. Annals of Mathematics. Second Series 48, pp. 441-471.
[21] Fuchs, M. and Seregin, G.: Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics, 1749. Springer-Verlag, Berlin, 2000. vi +269 pp .
[22] Gagliardo, E., 1957. Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in $n$ variabili. Rendiconti del seminario matematico della universita di Padova, 27, pp.284-305.
[23] Gmeineder, F. and Raiță, B., 2019. Embeddings for $\mathbb{A}$-weakly differentiable functions on domains. Journal of Functional Analysis, 277(12), 108278.
[24] Gmeineder, F., Raiță, B., and Van Schaftingen, J., 2021. On limiting trace inequalities for vectorial differential operators. Indiana Univ. Math. J. 70 (5), 2133-2176.
[25] Hernandez, F. and Spector, D., 2020. Fractional integration and optimal estimates for elliptic systems. arXiv preprint arXiv:2008.05639.
[26] Hörmander, L., 2003. The analysis of linear partial differential operators I: Distribution theory and Fourier analysis. Classics in Mathematics, Reprint of the 2nd Edition 1990. Springer, Berlin, Heidelberg.
[27] Hörmander, L., 1966. Pseudo-differential operators and non-elliptic boundary problems. Annals of Mathematics, pp.129-209.
[28] Kirchheim, B. and Kristensen, J., 2016. On rank one convex functions that are homogeneous of degree one. Archive for Rational Mechanics and Analysis, 221(1), pp.527-558.
[29] Korn, A., 1909. Über einige Ungleichungen, welche in der Theorie der elastischen und elektrischen Schwingungen eine Rolle spielen. Bulletin International de l'Académie des Sciences de Cracovie, pp. 705-724.
[30] Lions, J.-L. and Magenes, E., 1972. Non-homogeneous boundary value problems and applications Vol. I. Die Grundlehren der mathematischen Wissenschaften, Band 181. Springer-Verlag, New York-Heidelberg.
[31] LopatinskiĬ, Y.B., 1953. On a method of reducing boundary problems for a system of differential equations of elliptic type to regular integral equations. Ukrain. Mat. Ž, 5, pp.123-151.
[32] Mironescu, P., 2015. Note on Gagliardo's theorem "tr $\mathrm{W}^{1,1}=\mathrm{L}^{1} "$. In Ann. Univ. Buchar. Math. Ser. 6(LXIV) (1), 99-103.
[33] Mironescu, P. and Russ, E., 2015. Traces of weighted Sobolev spaces. Old and new. Nonlinear Analysis: Theory, Methods \& Applications, 119, pp. 354-381.
[34] Mityagin, B.S., 1958. On second mixed derivative. In Doklady Akademii Nauk (Vol. 123, No. 4, pp. 606-609). Russian Academy of Sciences.
[35] Ornstein, D., 1962. A non-inequality for differential operators in the $\mathrm{L}_{1}$ norm. Archive for Rational Mechanics and Analysis, 11(1), pp. 40-49.
[36] Peetre, J., 1979. A counterexample connected with Gagliardo's trace theorem. Comment. Math. Special Issue, 2, pp. 277-282.
[37] Pelczynski, A. and Wojciechowski, M., 2002. Sobolev spaces in several variables in L1-type norms are not isomorphic to Banach lattices. Arkiv för Matematik, 40(2), pp.363-382.
[38] Pelczynski, A. and Wojciechowski, M., 2003. Spaces of functions with bounded variation and Sobolev spaces without local unconditional structure. Journal für die Reine und Angewandte Mathematik, 558, pp.109-157
[39] RaițĂ, B., 2018. L¹-estimates for constant rank operators. arXiv preprint arXiv:1811.10057.
[40] RaițĂ, B., 2019. Critical $L^{p}$-differentiability of $B V^{\mathbb{A}}$-maps and canceling operators. Transactions of the American Mathematical Society, 372(10), pp.7297-7326.
[41] RaițĂ, B. and Skorobogatova, A., 2020. Continuity and canceling operators of order $n$ on $\mathbb{R}^{n}$. Calculus of Variations and Partial Differential Equations, 59(2), pp. 1-17.
[42] Smith, K.T., 1961. Inequalities for formally positive integro-differential forms. Bulletin of the American Mathematical Society, 67(4), pp. 368-370.
[43] Smith, K.T., 1970. Formulas to represent functions by their derivatives. Mathematische Annalen, 188(1), pp. 53-77.
[44] Stolyarov, D., 2020. Weakly canceling operators and singular integrals. ArXiv preprint arXiv:2006.11617.
[45] Spector, D. and Van Schaftingen, J., 2019. Optimal embeddings into Lorentz spaces for some vector differential operators via Gagliardo's lemma. Rendiconti Lincei-Matematica e Applicazioni, 30(3), pp.413-436.
[46] Stolyarov, D., 2020. Hardy-Littlewood-Sobolev inequality for $p=1$. ArXiv preprint arXiv:2010.05297.
[47] Strang, G. and Temam, R., 1981. Functions of bounded deformation. Archive for Rational Mechanics and Analysis, 75 (1981), pp. 7-21.
[48] Triebel, H., 1983: Theory of Function Spaces. In: Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel.
[49] UspenskiĬ, S.V., 1961. Imbedding theorems for classes with weights. Trudy Matematicheskogo Instituta imeni VA Steklova, 60, pp. 282-303.
[50] Van Schaftingen, J., 2004. Estimates for L1-vector fields. Comptes Rendus Mathematique, 339(3), pp. 181-186.
[51] Van Schaftingen, J., 2008. Estimates for L1-vector fields under higher-order differential conditions. Journal of the European Mathematical Society, 10(4), pp. 867-882.
[52] Van Schaftingen, J., 2010. Limiting fractional and Lorentz space estimates of differential forms. Proceedings of the American Mathematical Society, pp. 235-240.
[53] Van Schaftingen, J., 2013. Limiting Sobolev inequalities for vector fields and canceling linear differential operators. Journal of the European Mathematical Society, 15(3), 877-921.
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