A VERY QUICK INTRODUCTION TO REAL AND FUNCTIONAL ANALYSIS

FRANZ GMEINEDER

ABSTRACT. This document aims to serve as a blackbox for the seminar *Inequalities* in winter term 19/20. In particular, it comprises the foundations of measure theory and functional analysis as required in the upcoming talks. As such, it will be extended as the seminar evolves; if you spot any mistakes or feel that some sections should be extended, feel free to contact me.

CONTENTS

1. MEASURE AND INTEGRATION THEORY

1.1. **Signed and vector measures.** We assume the reader to be familiar with the concepts of Lebesgue integration, and thus focus exclusively on some more advanced topics. In this respect, we begin with the notions of *signed* and *vector measures*.

Let (X, Σ) be a measurable space. We say that $\mu\colon \Sigma \to \mathbb{R}^m$ is a *vector measure* or $\mathbb{R}^m$*-valued measure* provided

(a) $\mu(\emptyset) = 0$ and
(b) for all any sequence (A_j) of pairwise disjoint elements of Σ there holds

$$\mu\Big(\bigcup_j A_j\Big) = \sum_j \mu(A_j).$$

If $m = 1$, we say that μ satisfying the above requirements is a *signed measure*. Let us moreover arrange to call a set function $\mu\colon (X, \Sigma) \to [0, \infty]$, which satisfies (a) and (b) from above, a *positive measure*; it is finite if $\mu(X) < \infty$.

Vector measures on (X, Σ) form a vector space. This space can be normed by virtue of the *total variation*. For a vector measure μ on (X, Σ), put

$$|\mu|(A) := \sup\Big\{\sum_j |\mu(A_j)|\colon\ A_j \in \Sigma \text{ are pairwise disjoint with } A = \bigcup_j A_j\Big\}, \quad A \in \Sigma.$$

In this situation, $|\mu|$ is a positive, finite (!) measure on (X, Σ), too, making the space of $\mathbb{R}^m$-valued measures a normed vector space. It is sometimes useful to employ the structure of the underlying space X, allowing to introduce the important class of *Radon measures*.

For most of our applications, X will be a subset of $\mathbb{R}^n$ and thus is a metric space in itself. Hence let (X, d) be

(a) *locally compact*, i.e., for every $x \in X$ there exist an open set U and a compact set K such that $x \in U \subset K$.

(b) *separable*, i.e., there exists a countable dense subset of X.

As usual, the Borel σ-algebra (that is, the σ-algebra generated by the sets open for d) on X is denoted $\mathcal{B}(X)$. Henceforth, let (X,d) be locally compact and separable. We say that an $\mathbb{R}^m$-valued set function μ on X is

(a) *a Borel measure* if $m=1$ and μ is a positive measure on $(X,\mathcal{B}(X))$.
(b) *a Radon measure* if μ is defined on the relatively compact Borel subsets of X and is a measure on $(K,\mathcal{B}(K))$ for any compact set $K\subset X$. If, moreover, $\mu\colon \mathcal{B}(X)\to\mathbb{R}^m$ is a measure in the above sense, then we call μ a *finite Radon measure*.

The integration theory with respect to measures as introduced above follows the usual approach as known from the Lebesgue integral. The fundamental theorem we shall rely on then is given by the following result:

Theorem 1.1 (Riesz representation theorem for Radon measures*)**.** *Let $\Omega\subset\mathbb{R}^n$ be open. Then we have $\mathcal{M}(\Omega)\cong \mathrm{C}_0(\Omega)'$, the isometrical isomorphism being given by*

$$\mathcal{M}(\Omega)\ni\mu\mapsto\big(v\mapsto\int_\Omega v\,\mathrm{d}\mu\big)\in \mathrm{C}_0(\Omega)'.$$

Let us finally address the RADON-NIKODÝM THEOREM. Given a measurable space (X,Σ), a positive measure μ and an $\mathbb{R}^m$-valued measure ν on (X,Σ), we call ν *absolutely continuous for* μ provided

$$A\in\Sigma \text{ and } \mu(A)=0\Longrightarrow \nu(A)=0.$$

We then write $\nu\ll\mu$. Conversely, if there exists $A\in\Sigma$ such that $\mu(A)=0$ and $\nu(X\setminus A)=0$, then we call μ and ν *mutually singular* and write $\mu\perp\nu$.

Theorem 1.2 (Radon-Nikodým)**.** *In the above situation, assume that μ is σ-finite. Then there exists a unique pair (ν^a,ν^s) such that $\nu^a\ll\mu$, $\nu^s\perp\mu$ and $\nu=\nu^a+\nu^s$. Moreover, there exists a unique element $f\in\mathrm{L}^1(X;\mu;\mathbb{R}^m)$ such that $\nu^a=f\mu$. In this situation, we call f the* density of ν *with respect to μ and denote $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}:=f$.*

1.2. **Lebesgue and Hausdorff measures.** We write $\mathscr{L}^n$ for the n-dimensional Lebesgue measure. Let $s\geqslant 0$. Given $\delta\in(0,\infty]$ and $A\subset\mathbb{R}^n$, we put

$$\mathscr{H}^s_\delta(A):=\inf\Big\{\omega_s\sum_{j=1}^\infty r_j^s\colon\ A\subset\bigcup_{j=1}^\infty \mathrm{B}(x_j,r_j),\ r_j\leqslant\frac{\delta}{2}\Big\}.$$

Note that $\delta\mapsto\mathscr{H}^s_\delta(A)$ is non-decreasing as $\delta\searrow 0$. We thus may define

$$\mathscr{H}^s(A):=\sup_{\delta>0}\mathscr{H}^s_\delta(A)=\lim_{\delta\searrow 0}\mathscr{H}^s_\delta(A).$$

We call $\mathscr{H}^s$ the *s-dimensional Hausdorff measure*. Unless stated otherwise, we let $\Omega\subset\mathbb{R}^n$ be open and denote $\mathrm{L}^p(\Omega)$ the space of p-integrable functions $u\colon\Omega\to\mathbb{R}$ such that

$$\|u\|_{\mathrm{L}^p(\Omega)}:=\Big(\int_\Omega |u|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}:=\Big(\int_\Omega |u|^p\,\mathrm{d}\mathscr{L}^n\Big)^{\frac{1}{p}}<\infty.$$

Sometimes we shall also require Lebesgue spaces with respect to certain Hausdorff measures (and then use the notation $\mathrm{L}^p(\Sigma;\mathscr{H}^s)$ for some $\mathscr{H}^s$-measurable set Σ), but no confusions will arise from this.

1.3. **Elementary Inequalities.** We now turn to various inequalities which have been encountered in previous courses, and begin with *Hölder's inequality*: If $1 \leqslant p < \infty$, then for all $f \in \mathrm{L}^p(\mathbb{R}^n)$ and $g \in \mathrm{L}^{p'}(\mathbb{R}^n)$ (where $\frac{1}{p} + \frac{1}{p'} = 1$ so that $p' = \frac{p}{p-1}$) there holds

$$\|fg\|_{\mathrm{L}^1(\mathbb{R}^n)} \leqslant \|f\|_{\mathrm{L}^p(\mathbb{R}^n)} \|g\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}. \tag{1.1}$$

Also recall *Young's convolution inequality*: If $f \in \mathrm{L}^1(\mathbb{R}^n)$ and $g \in \mathrm{L}^p(\mathbb{R}^n)$, then $f * g \in \mathrm{L}^p(\mathbb{R}^n)$, too, together with

$$\|f * g\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|f\|_{\mathrm{L}^1(\mathbb{R}^n)} \|g\|_{\mathrm{L}^p(\mathbb{R}^n)}. \tag{1.2}$$

Lastly, recall *Jensen*'s inequality: If $\Omega \subset \mathbb{R}^n$ is $\mathscr{L}^n$-measurable, $f \colon \mathbb{R} \to \mathbb{R}$ is convex and $u \in \mathrm{L}^1(\Omega)$, then

$$f\Big(\fint_\Omega u \,\mathrm{d}x) \leqslant \fint_\Omega f(u) \,\mathrm{d}x.$$

1.4. **Smooth approximation.** Often, inequalities for weakly differentiable functions are firstly established for smooth functions and then *transferred* by an approximation process. In many instances, such an approximation procedure works as follows: Pick a radially symmetric function $\rho \in \mathrm{C}_c^\infty(\mathrm{B}(0,1);[0,1])$ with $\|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)} = 1$. We then define, for $\varepsilon > 0$, the ε-rescaled version of ρ by $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$. Given $u \in \mathrm{L}^p(\mathbb{R}^n)$ (where $1 \leqslant p < \infty$), we then consider the *mollification* $u_\varepsilon := \rho_\varepsilon * u$. We then have each of the following:

(a) $\|u_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$,
(b) $\|u - u_\varepsilon\|_{\mathrm{L}^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \searrow 0$.

If we wish to approximate a given $u \in \mathrm{L}^p(\Omega)$, $\Omega \subset \mathbb{R}^n$ now being bounded and measurable, then we may consider $v_\varepsilon := \rho_\varepsilon * (\mathbf{1}_\Omega u)$ to obtain $v_\varepsilon \to v$ in $\mathrm{L}^p(\mathbb{R}^n)$. These matters turn out a bit more subtle if we work in the Sobolev space context – see below for more detail.

2. CONCEPTS FROM FUNCTIONAL ANALYSIS

2.1. **Operators.** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed vector spaces and $T \colon X \to Y$ be a linear operator. We say that T is

(i) *bounded* if and only if there exists $C > 0$ such that $\|Tx\|_Y \leqslant C\|x\|_X$ holds for all $x \in X$,
(ii) *compact* if and only if T maps bounded sequences to relatively compact sequences.

If T is bounded, then we define its *operator norm* via

$$\|T\|_{X\to Y} := \sup_{x \in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X}.$$

For a linear operator T, boundedness is equivalent to continuity; moreover, note that if T is compact, then it is automatically bounded, hence continuous. Note that if the target space is scalar (so, depending on the application, $\mathbb{R}$ or $\mathbb{C}$), we call T a *linear functional*. The *(topological) dual* of X is given by

$$X' := \Big\{f \colon X \to \mathbb{R} \colon \sup_{\substack{x \in X,\\ \|x\|_X \leqslant 1}} |f(x)| < \infty\Big\}.$$

Note that X' in general is strictly smaller than the algebraic dual; moreover, X' is *always* a Banach space – regardless of whether X is a Banach space. Iteratively,

$$X'' := (X')', \quad X^{(n)} := (X^{(n-1)})'.$$

It is often favorable to have an explicit description of the duals of Banach spaces (see, e.g., the above Theorem 1.1). Such descriptions are often referred to as *Riesz-type* theorems. Essentially by use of the Hölder inequality, one has, e.g.,

$$(\mathrm{L}^p(\Omega))' \cong \mathrm{L}^{p'}(\Omega), \qquad 1 \leqslant p < \infty.$$

This has to be understood in the following way: For each $T \in (\mathrm{L}^p(\Omega))'$ there exists a unique $f \in \mathrm{L}^{p'}(\Omega)$ such that

$$T(g) = \int_\Omega fg \, \mathrm{d}x \qquad \text{for all } g \in \mathrm{L}^p(\Omega),$$

and the dual norm of T equals $\|f\|_{\mathrm{L}^{p'}(\Omega)}$. Other examples are $(\ell^p(\mathbb{N})' \cong \ell^{p'}(\mathbb{N})$ for if $1 \leqslant p < \infty$; in the case of infinite dimensional Hilbert spaces $\mathcal{H}$, we have $\mathcal{H}' \cong \mathcal{H}$. If, moreover, $\mathcal{H}$ is even a separable Hilbert space, then $\mathcal{H} \cong \mathcal{H}' \cong \ell^2(\mathbb{N})$.

2.2. **Duals, Double Duals and Reflexivity.** Let $(X, \|\cdot\|_X)$ be a normed space (over $\mathbb{R}$).

We now give an overview over different sorts of convergence for sequences in a Banach space X. Let $x, x_1, x_2, ... \in X$. We say that

- (x_k) converges *in the norm sense or strongly* to x and write $x_k \to x$ if $\|x - x_k\|_X \to 0$, $k \to \infty$.
- (x_k) converges *in the weak sense or weakly* to x and write $x_k \rightharpoonup x$ if $\langle f, x_k \rangle \to \langle f, x \rangle$ for all $f \in X'$.

On the dual space X' (which is a normed space in itself), these notions can be employed as well. However, it is convenient to have a terminology for *pointwise convergence*. Thus, given $f, f_1, f_2, ... \in X'$, we say that

- (f_k) converges *in the weak*-sense* to f and write $f_k \overset{*}{\rightharpoonup}$ if for all $x \in X$ there holds $\langle f_k, x \rangle \to \langle f, x \rangle$ as $k \to \infty$.

In consequence, these notions equally generalise to the double dual X''. Note that $\iota\colon X \hookrightarrow X''$ given by

$$\iota(x)\colon X' \ni f \mapsto f(x)$$

embeds X into X''. If this map ι moreover is surjective and isometric, then we call the Banach space X (note that X then necessarily is Banach!) *reflexive*. Examples of reflexive spaces include $\ell^p(\mathbb{N})$ and $\mathrm{L}^p(\Omega)$ for $1 < p < \infty$; in the borderline cases $p \in \{1, \infty\}$, neither $\ell^1, \ell^\infty, \mathrm{L}^1$ nor L^∞ is reflexive. Still, L^∞ canonically arises as the dual of L^1. Thus, by the Banach-Alaoglu-Bourbaki theorem, we do have some weak*-compactness results on L^∞. On L^1, the situation is much more subtle for $\mathrm{L}^1(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open is *not the dual of any normed space*. The latter fact is not trivial to prove; one option to do so is the KREIN-MILMAN theorem.

Reflexivity is sometimes misinterpreted as the mere requirement $X \cong X''$ - *this is wrong*. In fact, one can come up with spaces satisfying $X \cong X''$ (the so-called *James space*), yet fail to be reflexive. The key issue is that, while there might be *some* isometric isomorphism $\mathcal{J}\colon X \to X''$, we cannot assert that $\mathcal{J} = \iota$, the canonical embedding in general.

Moreover, reflexivity is a key concept for compactness results to be discussed next.

2.3. **Compactness.** Recall the Bolzano-Weierstraß theorem: If a sequence $(x_k) \subset \mathbb{R}^N$ is bounded (for some and hence, by equivalence of all norms on finite dimensional spaces, all norms), then we may extract a convergent subsequence. This result is easily seen to fail in infinite dimensional spaces: Think of the sequence (e_k) of unit vectors in $\ell^2(\mathbb{N})$. Equally, there might be closed and bounded sets of infinite dimensional normed spaces which fail to be compact. This fundamental issue is manifested in the following theorem attributable to RIESZ:

Theorem 2.1 (Characterisation of finite dimensionality a lá Riesz)**.** The following are equivalent for a real Banach space $(X, \|\cdot\|)$:

(a) $\dim(X) < \infty$.
(b) Every sequence $(x_k) \subset X$ which is bounded possesses a convergent subsequence $(x_{k(i)})$ (convergence being understood for the norm topology).
(c) The closed unit ball is compact.
(d) Every closed and bounded set is compact.

This motivates the question of *which compactness results survive at all*. In fact, a lot of statements persist when passing to weaker notions of convergence, in turn being manifested by the BANACH-ALAOGLU-BOURBAKI theorem:

Theorem 2.2 (Banach-Alaoglu-Bourbaki compactness theorem)**.** Let $(X, \|\cdot\|)$ be a normed, real and separable vector space. If $(f_j) \subset X'$ is bounded with respect to the norm on X', then there exists $f \in X'$ and a subsequence $(f_{j(i)}) \subset (f_j)$ such that $f_{j(i)} \overset{*}{\rightharpoonup} f$ as $j \to \infty$.

Therefore, still a powerful compactness result is available on *duals of normed spaces*. Now, if X is reflexive, we can essentially (i) realise X as a dual space and (ii) identify the weak*-convergence on $X'' \cong X$ with the weak convergence on X. This yields one direction of so-called EBERLEIN-SHMULYAN theorem:

Theorem 2.3 (Eberlein-Shmulyan characterisation of reflexive spaces)**.** The following are equivalent for a real Banach space $(X, \|\cdot\|)$:

(a) $(X, \|\cdot\|)$ is reflexive.
(b) Every sequence $(x_j) \subset X$ which is bounded for the norm on X possesses a subsequence $(x_{j(i)}) \subset (x_j)$ such that $x_{j(i)} \rightharpoonup x$ as $i \to \infty$ for some $x \in X$.

In view of the preceding two theorems, one can proceed as follows to gain compactness: Even though X might not arise as the dual of any other normed space, it is sometimes possible to embed X into a dual space Y', $X \hookrightarrow Y'$. One key instance of this procedure is to embed $\mathrm{L}^1(\Omega) \hookrightarrow \mathcal{M}(\Omega) \cong \mathrm{C}_0(\Omega)'$.

Lastly, let us note that *compactness results for the norm topology are usually very hard to be obtained*. An instance, however, that we shall come back to frequently is given by the ASCOLI-ARZELÁ theorem; it provides us with a characterisation of the (relatively) compact subsets of $\mathrm{C}(X)$ for compact X:

Theorem 2.4 (Ascoli-Arzelá theorem)**.** *Let (X, d) be a compact metric space and $\mathcal{C}$ be a subset of $\mathrm{C}(X)$, the latter being equipped with the supremum norm. Then $\mathcal{C}$ is relatively compact*[1] *in $\mathrm{C}(X)$ if and only if $\mathcal{C}$ is*

(a) pointwisely bounded*: For each $f \in \mathcal{C}$ there exists $c_f > 0$ such that $|f(x)| \leqslant c_f$ for all $x \in X$ and*
(b) equicontinuous*: For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{C}$ there holds*

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon.$$

[1]Meaning that the closure in $\mathrm{C}(X)$ is compact.

2.4. **Special Case: Hilbert spaces.** Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $\mathcal{H}$ be a vector space over $\mathbb{F}$. A map $\langle\cdot,\cdot\rangle\colon \mathcal{H}\times\mathcal{H}\to\mathbb{F}$ on $\mathcal{H}$ is called an *inner product* provided

- for each $y\in\mathcal{H}$, the partial map $x\mapsto\langle x,y\rangle$ is $\mathbb{F}$-linear,
- for all $x,y\in\mathcal{H}$ we have $\langle x,y\rangle=\overline{\langle y,x\rangle}$ (conjugate symmetry),
- for all $x\in\mathcal{H}$ we have $\langle x,x\rangle\geqslant 0$, with equality if and only if $x=0$ (positive definiteness).

Given an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$, we call $(\mathcal{H},\langle\cdot,\cdot\rangle)$ a *Prehilbert space*. Setting $\|x\|:=\sqrt{\langle x,x\rangle}$, $(\mathcal{H},\|\cdot\|)$ becomes a normed space; if it is Banach, then we say that $(\mathcal{H},\langle\cdot,\cdot\rangle)$ is a *Hilbert space*.

Compared with other Banach spaces Hilbert space feature several noteworthy properties, making them particularly easy to deal with. This metaprinciple is reflected by the following statements, where we confine ourselves to the real case.

(a) Every separable Hilbert space is isometrically isomorphic to $\ell^2(\mathbb{N})$. Namely, pick an orthonormal basis $(e_i)_{i\in\mathbb{N}}$; then, for each $x\in\mathcal{H}$ we have

$$x=\sum_{i\in\mathbb{N}}\langle e_i,x\rangle e_i\mapsto(\langle e_i,x\rangle)_{i\in\mathbb{N}}\in\ell^2(\mathbb{N}).$$

The *Fourier coefficient map* $\mathcal{H}\ni x\mapsto(\langle e_i,x\rangle)\in\ell^2(\mathbb{N})$ is a bijective isometry.

(b) In the Hilbert space case, the Riesz representation theorem takes a particularly simple form: If $\mathcal{H}$ is a real Hilbert space, then $\mathcal{H}\cong\mathcal{H}'$. More precisely, for every $f\in\mathcal{H}'$ there exists a unique $u_f\in\mathcal{H}$ such that $f(x)=\langle u_f,x\rangle$ holds for all $x\in\mathcal{H}$. The map $f\mapsto u$ is a bijective isometry.

(c) As a consequence of the previous item, all Hilbert spaces are *reflexive*.

(d) Here is a warning: Consider the Gel'fand triple $\mathrm{W}^{1,2}(\mathrm{B})\hookrightarrow\hookrightarrow\mathrm{L}^2(\mathrm{B})\hookrightarrow\mathrm{W}^{-1,2}(\mathrm{B})$. On the other hand, $\mathrm{W}^{1,2}(\mathrm{B})\cong\ell^2(\mathbb{N})\cong\mathrm{W}^{-1,2}(\mathrm{B})$, but this does not (!) imply the wrong statement that $\ell^2(\mathbb{N})$ is compactly embedded into $\ell^2(\mathbb{N})$.

3. Distributions and Sobolev spaces

3.1. **Test functions and Distributions.** Let $\Omega\subset\mathbb{R}^n$ be open. The linear space of *test functions*

$$\mathrm{C}_c^\infty(\Omega):=\{u\in\mathrm{C}^\infty(\Omega)\colon\ \mathrm{spt}(u)\ \text{compact}\}$$

is denoted $\mathcal{D}(\Omega)$ when being endowed with the following notion of convergence: Given $u,u_1,u_2,...\in\mathrm{C}_c^\infty(\Omega)$, we say that $u_j\to u$ in $\mathcal{D}(\Omega)$ if and only if there exists *a compact set* $K\subset\Omega$ such that

(a) $\mathrm{spt}(u),\mathrm{spt}(u_j)\subset K$ for all $j\in\mathbb{N}$ and
(b) $\|\partial^\alpha(u-u_j)\|_{\mathrm{L}^\infty(\Omega)}\to 0$ as $j\to\infty$ for all $\alpha\in\mathbb{N}_0^n$.

Consequently, the linear functionals which are continuous for this sort of convergence are called *distributions* on Ω and denote $\mathcal{D}'(\Omega)$. More precisely, we say that a linear map $T\colon\mathcal{D}(\Omega)\to\mathbb{R}$ is a distribution (in formulas $T\in\mathcal{D}'(\Omega)$) if

$$\langle T,\varphi_j\rangle\to\langle T,\varphi\rangle\qquad\text{whenever }\varphi_j\to\varphi\quad\text{in }\mathcal{D}(\Omega).$$

Here and in all of what follows, we have used the duality pairing notation $\langle T,\varphi\rangle:=T(\varphi)$ for $T\in\mathcal{D}'(\Omega)$ and $\varphi\in\mathcal{D}(\Omega)$. Some of the most important examples of distributions are as follows:

- $\mathrm{L}^1_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. Given $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$, we define a distribution $T_f \in \mathcal{D}'(\Omega)$ via
$$\langle T_f, \varphi \rangle := \int_\Omega f\varphi \,\mathrm{d}x, \qquad \varphi \in \mathcal{D}(\Omega).$$
We shall often abuse notation and simply write $\langle f, \varphi \rangle := \langle T_f, \varphi \rangle$. Distributions $T \in \mathcal{D}'(\Omega)$ which arise as $T = T_f$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ are called *regular distributions*. Note carefully that not every distribution is regular, as can be seen from the next item.
- $\mathcal{M}_{\mathrm{loc}}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. Given $\mu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$, we define a distribution $T_\mu \in \mathcal{D}'(\Omega)$ via
$$\langle T_\mu, \varphi \rangle := \int_\Omega \varphi \,\mathrm{d}\mu, \qquad \varphi \in \mathcal{D}(\Omega).$$
We shall often abuse notation and simply write $\langle \mu, \varphi \rangle := \langle T_\mu, \varphi \rangle$. Distributions which are as $T = T_\mu$ are called *measure regular distributions*.

Distributions can be differentiated, the definition of the distributional derivative being motivated by the usual integration-by-parts formula. Given $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$, we define a new distribution $\partial^\alpha T \in \mathcal{D}'(\Omega)$ by its action on $\varphi \in \mathcal{D}(\Omega)$ via
$$\langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle.$$

3.2. **Schwartz functions and tempered distributions.** Many operators which are originally defined on $\mathrm{C}_c^\infty(\mathbb{R}^n)$ do not map into $\mathrm{C}_c^\infty(\mathbb{R}^n)$, an instance being given by the Fourier transform. The Fourier transform, however, turns out bijective on the Schwartz space (or class) to be recalled next. For $x \in \mathbb{R}^n$, define $\langle x \rangle := \sqrt{1 + |x|^2}$. We say that a function $u\colon \mathbb{R}^n \to \mathbb{R}$ belongs to the *Schwartz class* $\mathcal{S}(\mathbb{R}^n)$ provided $u \in \mathrm{C}^\infty(\mathbb{R}^n)$ and for any $l, m \in \mathbb{N}_0$ there holds
$$[u]_{l,m} := \sup_{|\alpha| \leqslant m} \sup_{x \in \mathbb{R}^n} \langle x \rangle^l |\partial^\alpha u(x)| < \infty.$$
Let $u, u_1, u_2, ... \in \mathcal{S}(\mathbb{R}^n)$. We say that (u_j) converges to u in $\mathcal{S}(\mathbb{R}^n)$ provided $[u - u_j]_{l,m} \to 0$ as $j \to \infty$ for all $l, m \in \mathbb{N}_0$. This convergence arises from the metric
$$d_{\mathcal{S}}(u, v) := \sum_{l,m \in \mathbb{N}_0} c_{l,m} \frac{[u-v]_{l,m}}{1 + [u-v]_{l,m}}.$$
A crucial operator in all of what follows is the *Fourier transform*. Given $u \in \mathcal{S}(\mathbb{R}^n)$, we define
$$\mathcal{F}u[\xi] := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-\mathrm{i}\, x \cdot \xi} \,\mathrm{d}x, \qquad \xi \in \mathbb{R}^n. \tag{3.1}$$

Theorem 3.1 (Fourier inversion)**.** *The Fourier transform $\mathcal{F}\colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is bijective. Moreover, both $\mathcal{F}$ and the inverse map $\mathcal{F}^{-1}$ are continuous on $\mathcal{S}(\mathbb{R}^n)$.*

Lastly, the Fourier transform extend to a bounded linear operator $\mathcal{F}\colon\ \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$. This operator is a bijective L^2-isometry.

We then define the space of *tempered distributions* by
$$\mathcal{S}'(\mathbb{R}^n) := \big\{ T\colon \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}\colon\ T \text{ linear and continuous for convergence in } \mathcal{S}(\mathbb{R}^n) \big\}.$$
Canonically, given $T, T_1, T_2, ... \in \mathcal{S}(\mathbb{R}^n)$, we say that $T \overset{*}{\rightharpoonup} T$ provided $\langle T_j, \varphi \rangle \to \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Note that *every tempered distribution is a distribution but not vice versa*; an example is given by the shifted 'comb'
$$T = \sum_{n \in \mathbb{N}} \delta^{(n)}(\cdot - n).$$

We lastly intend to declare the Fourier transform on tempered distributions. This is approached via duality; we put for $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\langle \mathcal{F}T, \varphi \rangle := \langle T, \mathcal{F}\varphi \rangle. \tag{3.2}$$

For various parts of the seminar, it is convenient to have the following result on radial, homogeneous distributions. Note that, if $-n < a < 0$, then

$$S_a\varphi \mapsto \int_{\mathbb{R}^n} |x|^a \varphi(x)\,\mathrm{d}x, \qquad \varphi \in \mathcal{S}(\mathbb{R}^n) \tag{3.3}$$

gives rise to a tempered distribution. We then have

Theorem 3.2 (Homogeneous, radial tempered distributions)**.** *Given $-n < a < 0$, define S_a by* (3.3)*. Then the Fourier transform of S_a is given by the tempered distribution induced by $\xi \mapsto C_{n,a}|\xi|^{-n-a}$, where $C_{n,a} > 0$ is a constant.*

3.3. **Sobolev spaces.** Let $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. For an open set $\Omega \subset \mathbb{R}^n$, we introduce the *Sobolev space* $\mathrm{W}^{k,p}(\Omega)$ as the collection of $u \in \mathrm{L}^p(\Omega)$ such that

all distributional partial derivatives up to order k can be represented by L^p-functions.

In particular, the distributional derivatives are regular distributions, and we briefly write

$$\mathrm{W}^{k,p}(\Omega) := \Big\{u \in \mathrm{L}^p(\Omega)\colon\ \|u\|_{\mathrm{W}^{k,p}(\Omega)} := \Big(\sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{\mathrm{L}^p(\Omega)}^p\Big)^{\frac{1}{p}} < \infty\Big\}.$$

When endowed with $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$, $\mathrm{W}^{k,p}(\Omega)$ is a Banach space. If $p < \infty$, it is convenient to moreover introduce the natural subspace of Sobolev functions with zero boundary values, $\mathrm{W}_0^{k,p}(\Omega)$, as the completion of $\mathrm{C}_c^\infty(\Omega)$ for $\|\cdot\|_{\mathrm{W}^{k,p}(\Omega)}$. We omitted the case $p = \infty$ for the following reason. If Ω has, say, smooth boundary, then $\mathrm{W}^{1,\infty}(\Omega)$ coincides with the Lipschitz functions $\mathrm{C}^{0,1}(\Omega)$. As such, we ought to define $\mathrm{W}_0^{1,\infty}(\Omega)$ as the Lipschitz functions vanishing at the boundary $\partial\Omega$. Note that the closure of $\mathrm{C}_c^\infty(\Omega)$ for the $\mathrm{W}^{1,\infty}$-norm is just $\mathrm{C}_0^1(\Omega)$, and this space is strictly smaller than $\mathrm{C}^{0,1}(\Omega)$.

If $p = 2$ and $\Omega = \mathbb{R}^n$, then the Sobolev spaces $\mathrm{W}^{k,2}(\mathbb{R}^n)$ can be characterised by the Fourier transform. The idea is this: For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have $\mathcal{F}(\partial^\alpha\varphi)(\xi) = (-1)^{|\alpha|}\xi^\alpha\mathcal{F}\varphi(\xi)$ for all $\xi \in \mathbb{R}^n$. Now, by Plancherel's theorem (i.e., $\mathcal{F}\colon\ \mathrm{L}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$ isometrically),

$$\begin{aligned}\Big(\sum_{|\alpha|\leqslant k} \|\partial^\alpha u\|_{\mathrm{L}^2(\mathbb{R}^n)}^2\Big)^{\frac{1}{2}} &= \Big(\sum_{|\alpha|\leqslant k} \|\mathcal{F}(\partial^\alpha u)\|_{\mathrm{L}^2(\mathbb{R}^n)}^2\Big)^{\frac{1}{2}} \\ &= \Big(\int_{\mathbb{R}^n}\Big(\sum_{|\alpha|\leqslant k} |\xi|^{2|\alpha|}\Big)|\mathcal{F}\varphi(\xi)|^2\,\mathrm{d}\xi\Big)^{\frac{1}{2}}.\end{aligned}$$

Put $p_k(\xi) := \sum_{|\alpha|\leqslant k} |\xi|^{2|\alpha|}$ and $\langle\xi\rangle := \sqrt{1+|\xi|^2}$. Then there holds $p_k(\xi) \simeq \langle\xi\rangle^{2k}$ and hence $\|u\|_{\mathrm{W}^{k,2}(\mathbb{R}^n)} \simeq \|\langle\cdot\rangle^k\mathcal{F}u\|_{\mathrm{L}^2(\mathbb{R}^n)}$. In fact, we have for $u \in \mathrm{L}^2(\mathbb{R}^n)$:

$$u \in \mathrm{W}^{k,2}(\mathbb{R}^n) \Longleftrightarrow \|\langle\cdot\rangle^k\mathcal{F}u\|_{\mathrm{L}^2(\mathbb{R}^n)} < \infty.$$

More generally, one has the following equivalent characterisations of $\mathrm{W}^{1,p}(\mathbb{R}^n)$:

Theorem 3.3. *Let $1 < p < \infty$. Then the following are equivalent for $u \in \mathrm{L}^p(\mathbb{R}^n)$:*

(a) $u \in \mathrm{W}^{1,p}(\mathbb{R}^n)$.

(b) *There exists $c > 0$ such that for any $s \in \{1, ..., n\}$ and any $h > 0$ we have*

$$\|\tfrac{1}{h}(u(\cdot + he_s) - u(\cdot)\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant c.$$

If $p = 2$*, then* (a) *and* (b) *from above are moreover equivalent to*

(c) $\|\langle\cdot\rangle\mathcal{F}u\|_{\mathrm{L}^2(\mathbb{R}^n)} < \infty$.

For most of the seminar, the following approximation result is instrumental:

Theorem 3.4 (Smooth approximation in Sobolev spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be open. Then, for each $1 \leqslant p < \infty$ and every $k \in \mathbb{N}$, the space $(\mathrm{C}^\infty \cap \mathrm{W}^{k,p})(\Omega)$ is dense in $\mathrm{W}^{k,p}(\Omega)$ for the norm topology.

The proof works by *localised smoothing*. Next we discuss the possibility of extending and assigning boundary values to Sobolev functions:

Theorem 3.5 (Extension and trace theorem)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with Lipschitz boundary* $\partial\Omega$*. Then*

(a) *there exists a bounded linear extension operator* $E\colon\ \mathrm{W}^{1,p}(\Omega) \to \mathrm{W}^{1,p}(\mathbb{R}^n)$*, i.e.,* E *is linear and bounded and satisfies* $Eu|_\Omega = u$ *in* $\mathrm{W}^{1,p}(\Omega)$ *for all* $u \in \mathrm{W}^{1,p}(\Omega)$.

(b) *there exists a bounded linear trace operator* $\mathrm{Tr}\colon\ \mathrm{W}^{1,p}(\Omega) \to \mathrm{L}^p(\partial\Omega; \mathscr{H}^{n-1})$*, i.e.,* Tr *is linear and bounded and we have* $\mathrm{Tr}(u) = u|_{\partial\Omega}$ *for all* $u \in \mathrm{W}^{1,p}(\Omega) \cap \mathrm{C}(\overline{\Omega})$.

Note that the last part of the theorem must be approached with care: Even if we extend a Sobolev function to the entire $\mathbb{R}^n$, we cannot simply restrict the extension to $\partial\Omega$ without further comments (note that $\partial\Omega$ is a nullset for $\mathscr{L}^n$). Also note that, if $1 < p < \infty$, then the trace operator is *not onto* $\mathrm{L}^p(\partial\Omega; \mathscr{H}^{n-1})$.

On the other hand, by the Arzelá-Ascoli theorem, mollification and a diagonal argument one has

Theorem 3.6 (Rellich-Kondrachov compactness theorem*)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded with Lipschitz boundary* $\partial\Omega$*. Then for any* $1 \leqslant p < \infty$ *we have the compact embedding*

$$\mathrm{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow \mathrm{L}^p(\Omega).$$

3.4. **Negative Sobolev spaces*.** For $1 < p < \infty$, $k \in \mathbb{N}$ and an open subset Ω of $\mathbb{R}^n$, we define

$$\mathrm{W}^{-k,p}(\Omega) := (\mathrm{W}_0^{k,p'}(\Omega))'$$

and equip $\mathrm{W}^{-k,p}$ with the usual dual norm. To get some better understanding of these spaces, we make use of the following heuristics: If $u \in \mathrm{W}^{k,p}(\Omega)$, then we may differentiate u k-times to obtain an L^p-function. So, if $u \in \mathrm{W}^{-k,p}(\Omega)$, we expect that a k-fold integration (!) yields an L^p-function. Thus, we expect elements of $\mathrm{W}^{-k,p}$ to be the k-fold derivatives of L^p-functions. Here is a sample theorem:

Theorem 3.7. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded. A linear functional* $\Phi\colon\ \mathrm{W}^{1,2}(\Omega) \to \mathbb{R}$ *belongs to* $\mathrm{W}^{-1,2}(\Omega)$ *if and only if there exist* $f_0, f_1, ..., f_n$ *such that*

$$\Phi = f_0 - \sum_{j=1}^{n} \partial_j f_j \qquad \text{in } \mathcal{D}'(\Omega).$$

Moreover,

$$\|\Phi\|_{\mathrm{W}^{-1,2}(\Omega)} = \inf\Big\{\Big(\sum_{j=0}^{\infty} \|f_j\|_{\mathrm{L}^2(\Omega)}^2\Big)^{\frac{1}{2}} :\ \Phi = f_0 - \sum_{j=1}^{n} f_j \text{ in } \mathcal{D}'(\Omega) \text{ for } f_0, ..., f_n \in \mathrm{L}^2(\Omega)\Big\}.$$

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