PARTIAL REGULARITY FOR
SYMMETRIC QUASICONVEX FUNCTIONALS ON BD

FRANZ GMEINEDER

ABSTRACT. We establish the first partial regularity results for (strongly) symmetric quasiconvex functionals of linear growth on BD, the space of functions of bounded deformation. By RINDLER's foundational work [65], symmetric quasiconvexity is the pivotal notion as regards sequential weak*-lower semicontinuity and hence for the existence of minima of the relaxed functionals on BD. The overarching main difficulty here is the lack of KORN's Inequality in the L^1-setting, hereby implying that the BD-case is genuinely different from the study of variational integrals on BV. Unlike for superlinear growth, symmetric quasiconvex functionals, where we establish partial regularity by direct reduction to the full gradient case by KORN-type inequalities, such a reduction does not work in the linear growth case and identifies the latter as the only situation requiring a treatment on its own.

RÉSUMÉ. Nous établissons les premiers résultats de régularité partielles pour des fonctionnelles quasi-convexes (fortement) symétriques ayant une croissance linéaire sur BD, l'espace des fonctions dont la déformation est bornée. De part les travaux précurseurs de RINDLER [65], la quasi-convexité symétrique est la notion centrale relativement à la semi-continuité inférieure faible-* et donc pour l'existence des minima de fonctionnelles relaxées sur BD. La difficulté générale ici est due à l'absence de l'inégalité de Korn dans le cadre fonctionel L^1, qui de ce fait révèle une différence fondamentale entre le cas BD et celui relatif à l'étude des intégrales variationnelles sur BV. Contrairement aux fonctionnelles quasi-convexes symétriques à croissance superlinéaire pour lesquelles nous établissons la régularité partielle via réduction directe au cas du gradient complet grâce aux inégalités de type Korn, une telle réduction ne peut être implémentée dans le cas de la croissance linéaire et identifie ce dernier comme étant la seule situation qui nécéssite un traitement particulier.

CONTENTS

1. INTRODUCTION

1.1. **Aims and scope.** Let $n \geq 2$ and Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary. A vast class of variational problems connected to plasticity is set up by virtue of linear growth functionals depending on the symmetric gradient, cf. [9, 40, 71, 20]. Possibly allowing

for non-convex energies, a unifying perspective on the topic as considered in variants in [12, 65] is given by the canonical variational principle

$$(1.1) \qquad \text{to minimise} \quad F[v] := \int_{\Omega} f(\varepsilon(v)) \, dx \quad \text{over a Dirichlet class } \mathscr{D}_{u_0},$$

where $u_0 \colon \Omega \to \mathbb{R}^n$ is a suitable Dirichlet datum and $\varepsilon(v) := \frac{1}{2}(\mathrm{D}v + \mathrm{D}v^{\top})$ denotes the *symmetric gradient* of a map $v \colon \mathbb{R}^n \to \mathbb{R}^n$. Most crucially, $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ is assumed to be a continuous integrand of *linear growth*. By this we understand that there exists a constant $L > 0$ such that

$$(\text{LG}) \qquad\qquad |f(z)| \le L(1 + |z|) \qquad \text{for all } z \in \mathbb{R}^{n \times n}_{\mathrm{sym}}.$$

Following the foundational work of RINDLER [65], a necessary and sufficient condition for the associated relaxed functionals to be suitably lower semicontinuous is that of *symmetric quasiconvexity*, cf. Section 1.2 below. In view of the direct method of the calculus of variations, symmetric quasiconvexity thus plays the central rôle for functionals of the form (1.1). Yet, for such symmetric quasiconvex functionals the properties of minima are far from being understood – in particular, a regularity theory is still missing. This equally applies to the situation where $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ is of *p*-growth ($|f(z)| \le L(1 + |z|^p)$ for all $z \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$) with $p > 1$, thereby connecting to models from nonlinear elasticity or fluid mechanics [25, 40]. Hence the objective of this paper is to make a first step in this direction and close this gap. As we shall see (cf. Section 1.4), it is only the linear growth case which requires a separate theory; the *p*-growth case with $p > 1$ can be fully reduced to the regularity theory for *full gradient functionals*.

To elaborate more on these matters, we start by noting that the growth bound (LG) suggests to consider (1.1) on Dirichlet classes $\mathrm{W}^{1,1}_{u_0}(\Omega; \mathbb{R}^n) =: u_0 + \mathrm{W}^{1,1}_0(\Omega; \mathbb{R}^n)$ for $u_0 \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^n)$. However, by ORNSTEIN's Non-Inequality [63], it is not possible to uniformly bound the L^1-norm of $\mathrm{D}u$ against that of $\varepsilon(u)$. In fact, for every $n \ge 2$ there exists a sequence $(\varphi_j) \subset \mathrm{C}^{\infty}_c(\mathrm{B}(0,1); \mathbb{R}^n)$ for which $(\varepsilon(\varphi_j))$ remains bounded in $\mathrm{L}^1(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}})$ whereas $\|\mathrm{D}\varphi_j\|_{\mathrm{L}^1(\Omega; \mathbb{R}^{n \times n})} \to \infty$ as $j \to \infty$, cf. CONTI et al. [24] and KIRCHHEIM & KRISTENSEN [49] for recent approaches. This is in stark contrast with the situation when L^1 is replaced by L^p, $1 < p < \infty$. Indeed, in the latter case the corresponding result can be reduced to KORN-type inequalities, cf. [39, 48, 61, 62] for classical material and CIARLET et al. [21, 22, 23] and the references therein for a selection of nonlinear variants. In consequence, F given by (1.1) is not coercive on $\mathrm{W}^{1,1}(\Omega; \mathbb{R}^n)$ but on

$$\mathrm{LD}(\Omega) := \left\{ u \in \mathrm{L}^1(\Omega; \mathbb{R}^n) \colon \varepsilon(u) \in \mathrm{L}^1(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}}) \right\}$$

endowed with the LD-norm $\|u\|_{\mathrm{LD}} := \|u\|_{\mathrm{L}^1} + \|\varepsilon(u)\|_{\mathrm{L}^1}$. It is thus natural to let the Dirichlet datum u_0 belong to $\mathrm{LD}(\Omega)$ and consider the variational principle (1.1) over the affine class $\mathscr{D}_{u_0} := \mathrm{LD}_{u_0}(\Omega) := u_0 + \mathrm{LD}_0(\Omega)$, where $\mathrm{LD}_0(\Omega)$ is the closure of $\mathrm{C}^1_c(\Omega; \mathbb{R}^n)$ for the LD-norm. Still, LD is an L^1-based space and hence fails to be reflexive; as a consequence, it lacks an appropriate version of the Banach-Alaoglu theorem concerning weak convergence. Thus it is required to relax F to the space $\mathrm{BD}(\Omega)$ given by

$$\mathrm{BD}(\Omega) := \left\{ u \in \mathrm{L}^1(\Omega; \mathbb{R}^n) \colon \mathrm{E}u \in \mathscr{M}(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}}) \right\}.$$

Here, $\mathscr{M}(\Omega; \mathbb{R}^{n \times n}_{\mathrm{sym}})$ are the finite, $\mathbb{R}^{n \times n}_{\mathrm{sym}}$-valued Radon measures on Ω, and we use the notation $\mathrm{E}u$ instead of $\varepsilon(u)$ to indicate that $\mathrm{E}u$ is a measure. The relaxation here is taken with respect to weak*-convergence in $\mathrm{BD}(\Omega)$, and we refer the reader to Sections 1.2 and 2.2 for the requisite background terminology. This space – which contains $\mathrm{BV}(\Omega; \mathbb{R}^n)$ as a proper subspace – takes a prominent role in plasticity, and has been studied from various perspectives by a notable plenty of authors, see [71, 20, 50, 51, 4, 8, 69, 11] among others. Before embarking on the regularity issue raised above in detail, we briefly pause and discuss the relevant relaxed functionals $\overline{F}$ that are required for defining the notion of (local) minimality in the sequel.

1.2. **Symmetric quasiconvexity and relaxation.** From a calculus of variations and hereafter lower semicontinuity perspective, the central notion for functionals F is a variant of MORREY's *quasiconvexity* [60], namely the aforementioned *symmetric quasiconvexity*. Already appearing in variants in [26, 38], we start by recalling the following definition as given, e.g., in [12, 65]:

Definition 1.1 (Symmetric quasiconvexity). A continuous integrand $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ is said to be *symmetric quasiconvex* provided there holds

$$f(z) \leq \int_Q f(z + \varepsilon(\varphi)) \, \mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}^1_0(Q; \mathbb{R}^n) \text{ and } z \in \mathbb{R}^{n \times n}_{\mathrm{sym}},$$

where $Q = (0,1)^n$ is the open unit cube in $\mathbb{R}^n$.

Letting $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$, we denote $\mathrm{E}u = \mathrm{E}^a u + \mathrm{E}^s u = \mathscr{E}u\mathscr{L}^n + \frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|}|\mathrm{E}^s u|$ the Lebesgue-Radon-Nikodým decomposition of $\mathrm{E}u$; cf. Section 2.2 for this and the subsequent terminology. Returning to the functional F defined in terms of $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ by (1.1) subject to (LG), let $v \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ be given. For a subset $\omega \subseteq \Omega$ with Lipschitz boundary $\partial \omega$, we define the *relaxed functional* by

$$(1.2) \quad \begin{aligned} \overline{F}_v[u; \omega] = {} & \int_\omega f(\mathscr{E}u) \, \mathrm{d}x + \int_\omega f^\infty\Big(\frac{\mathrm{d}\mathrm{E}u}{\mathrm{d}|\mathrm{E}u|}\Big) \mathrm{d}|\mathrm{E}^s u| \\ & + \int_{\partial \omega} f^\infty(\mathrm{Tr}_{\partial \omega}(v - u) \odot \nu_{\partial \omega}) \, \mathrm{d}\mathscr{H}^{n-1}, \qquad u \in \mathrm{BD}(\omega). \end{aligned}$$

Here, $f^\infty(z) := \limsup_{t \searrow 0} t f(\frac{z}{t})$ denotes the recession function of f at $z \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$, capturing the integrand's behaviour at infinity. Also, $\mathrm{Tr}_{\partial \omega}$ displays the boundary trace operator on $\mathrm{BD}(\omega)$ and $\nu_{\partial \omega}$ the outer unit normal to $\partial \omega$.

With this notation, we say that $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ is a *local* BD-*minimiser* (or *local generalised minimiser*) for F provided

$$(1.3) \qquad \overline{F}_u[u; \omega] \leq \overline{F}_u[v; \omega] \qquad \text{for all } v \in \mathrm{BD}(\omega)$$

holds for all subsets $\omega \Subset \Omega$ with Lipschitz boundary $\partial \omega$. If $u_0 \in \mathrm{LD}(\Omega)$ is a given Dirichlet datum, we put $\overline{F}_{u_0}[u] := \overline{F}_{u_0}[u; \Omega]$. In this situation, we call $u \in \mathrm{BD}(\Omega)$ a BD-*minimiser* (or *generalised minimiser*) for F subject to u_0 provided

$$(1.4) \qquad \overline{F}_{u_0}[u] \leq \overline{F}_{u_0}[v] \qquad \text{for all } v \in \mathrm{BD}(\Omega).$$

Then, any BD-minimiser subject to u_0 is a local BD-minimiser. Essentially solely subject to the additional linear growth assumption (LG), RINDLER [65] identified the symmetric quasiconvexity of f as a necessary and sufficient condition for $\overline{F}_{u_0}$ to be sequentially weak*-lower semicontinuous on $\mathrm{BD}(\Omega)$. Most notably, not only the extending the classical work of AMBROSIO & DAL MASO [6] as well as partly that of FONSECA & MÜLLER [37] from the BV- to the BD-situation, the relevant lower semicontinuity was established in [65] without relying on the BD-variant of ALBERTI's rank-one theorem [5]. By now, the latter has been proved by DE PHILIPPIS & RINDLER in the seminal work [28] in a much more general context, allowing for a simplified proof of (1.2) (cf. [29, 10, 17]) but had not been available at the time of [65].

In consequence, augmenting the linear growth assumption (LG) with a suitable coerciveness condition on the symmetric quasiconvex integrand f, existence of BD-minima for F follows at ease. Equally, we have the *no-gap-result* $\inf_{u_0 + \mathrm{LD}_0(\Omega)} F = \min_{\mathrm{BD}(\Omega)} \overline{F}_{u_0}$, see Section 6.1. As will be discussed below in Section 1.4, such a coerciveness criterion goes hand in hand with the partial regularity of BD-minima. It is thus natural to contextualise the partial regularity for BD-minima with available results in the literature and thereby outline the main obstructions first.

1.3. **Contextualisation and overview.** In the common language of regularity theory, the minimisation of functionals (1.1) displays a *purely vectorial problem*. Even in the case where the symmetric gradient in (1.1) is replaced by the full gradient, it is a well-known feature of such

multiple integrals to produce minima which are not everywhere $C^{1,\alpha}$-Hölder continuous but only on a large set; cf. [13, 42, 59] for overviews. This is referred to as *partial regularity*.

In the superlinear growth regime with full gradients, the study of partial regularity for minima has a long and rich history, starting with the seminal work of EVANS [36] and ACERBI & FUSCO [2]; also see MINGIONE et al. [19, 55, 59] and [13, 31, 32, 33, 34, 42] for a non-exhaustive list of other contributions. However, until recently, for full gradient linear growth functionals the only contribution had been the local-in-phase-space result due to ANZELLOTTI & GIAQUINTA [9] and its adaptation to the model integrands $z \mapsto (1 + |z|^p)^{1/p}$, $p \neq 2$, by SCHMIDT [66]. This approach, which crucially relies on comparison with mollifications and thus works well for convex integrands by Jensen's inequality, has been extended to the BD-setting by the author [44]. Yet, due to the very method of proof, it seems to be restricted to convex integrands and a generalisation of the strategy to the quasiconvex case seems difficult; also see the discussion in [9, 66] and [44].

At present, in the full gradient, quasiconvex linear growth case on BV, the only partial regularity result up to date has been given recently by KRISTENSEN and the author [46]. In this work, a direct comparison with suitably $\mathbb{A}$-harmonic maps is implemented that overcomes any indirect argument as is found e.g. in the blow-up method or, quite implicitly, in the proof of the $\mathbb{A}$-harmonic approximation lemma due to DUZAAR et al. [32, 33, 34]. Let us note that similarly to [9, 66, 44], the sole use of direct arguments is somehow dictated here by the comparatively weak compactness properties of BV and BD. In fact, examplarily pursuing the blow-up method for linear growth functionals, it is necessary to establish that the weak*-limit of a blow-up sequence satisfies a strongly elliptic Legendre-Hadamard system. However, by possible concentration effects, this conclusion seems unreachable since there are no general compactness improvements for the relevant blow-up sequences: Such compactness boosts would require some uniform local integrability enhancements, usually provided by GEHRING's lemma in reliance on Caccioppoli-type inequalites, or higher (fractional) differentiability estimates a lá MINGIONE [57, 58]. Whereas the former cannot be implemented in the linear growth situation – essentially due to the non-availability of a sublinear Sobolev-Poincaré-type inequality, cf. BUCKLEY & KOSKELA [18] and the discussion in Section 5.5 –, higher fractional differentiability results on minima such as in [57, 58] are confined to the convex situation. Similar issues already arise in the full gradient situation, equally for other methods such as the classical $\mathbb{A}$-harmonic approximation, and we refer the reader to [46] for a further discussion thereof.

1.4. **Main Results.** After these preparations, we now pass to the description of the main results of the present paper. To begin with, symmetric quasiconvexity and the linear growth hypothesis (LG) together are easily seen not to be sufficient for F given by (1.1) to produce bounded minimising sequences in $LD(\Omega)$. To ensure the latter, we require a strong version of symmetric quasiconvexity. The same issue arises in the superlinear growth regime as well, and so we begin with the treatment of such functionals, in turn being linked to elasticity type problems. Our first result is that – completely different from the linear growth situation – *in the superlinear growth case, partial regularity directly can be fully reduced to the corresponding full gradient theory*.

Given $1 \leq p < \infty$, we put $V_p(z) := (1 + |z|^2)^{\frac{p}{2}} - 1$ for $z \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and put $V(z) = V_1(z) = \sqrt{1 + |z|^2} - 1$. We say that $g \in C(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ is of *p-growth* provided there exists $L_p > 0$ such that

$$(1.5) \qquad\qquad |g(z)| \leq L_p(1 + |z|^p) \qquad \text{for all } z \in \mathbb{R}^{n \times n}_{\mathrm{sym}},$$

and accordingly call *g p-strongly symmetric quasiconvex* provided there exists $\ell_p > 0$ such that

$$(1.6) \qquad \mathbb{R}^{n \times n}_{\mathrm{sym}} \ni z \mapsto g(z) - \ell_p V_p(z) \in \mathbb{R} \qquad \text{is symmetric quasiconvex.}$$

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $1 < p < \infty$ and suppose that $g \in C^2(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ is an integrand which*

(a) *is of p-growth, i.e., satisfies (1.5) for all $z \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and*

(b) *is p-strongly symmetric quasiconvex in the sense of* (1.6).

Then for any local minimiser $u \in W^{1,p}_{loc}(\Omega; \mathbb{R}^n)$ of the corresponding integral functional

$$(1.7) \qquad v \mapsto \int g(\varepsilon(v)) \, dx$$

there exists an open subset Ω_u of Ω such that $\mathcal{L}^n(\Omega \setminus \Omega_u) = 0$ and u is of class $C^{1,\alpha}$ for each $0 < \alpha < 1$ in a neighbourhood of any of the elements of Ω_u.

Theorem 1.2 displays a sample theorem; similar results can be inferred for variational integrals with integrands $f(x, u, \varepsilon(u))$. In particular, the partial $C^{1,\alpha}$-regularity of minima of *convex* elasticity-type functionals (for some elliptic $\mathbb{C} \in \mathscr{L}(\mathbb{R}^{n \times n}_{sym}, \mathbb{R}^{n \times n}_{sym})$, $\mu > 0$ and $g \in L^p(\Omega; \mathbb{R}^n)$)

$$(1.8) \qquad v \mapsto \int_\Omega \frac{1}{p} \left(\left(\langle \mathbb{C}\varepsilon(v), \varepsilon(v) \rangle + \mu \right)^{\frac{p}{2}} - \mu^{\frac{p}{2}} \right) dx + \int_\Omega |v - g|^p \, dx,$$

as recently considered in [25] can similarly be covered and generalised by the methods underlying Theorem 1.2, even for any elliptic operator $A[D]$ in the sense of Section 2.2, and we shall pursue this elsewhere. The reason for the reducibility of Theorem 1.2 to the full gradient case is that the p-strong symmetric quasiconvexity (1.6) expresses a coerciveness property of the associated variational integrals on $W^{1,p}$. For $1 < p < \infty$ and subject to (1.6), minimising sequences can be proven to remain bounded in $W^{1,p}(\Omega; \mathbb{R}^n)$ by KORN-type inequalities. The p-strong symmetric quasiconvexity, being an integral rather than a pointwise condition, gives us direct access to the requisite KORN-type inequalities; see Section 3 for more detail.

If $p = 1$, ORNSTEIN's Non-Inequality [63, 24, 49] does not allow to employ a similar reduction scheme. More systematically, if $p = 1$ and (LG) are in action, the (1-)strong symmetric quasiconvexity still expresses a coerciveness property for the associated variational integrals on LD (see Lemma 2.7). To streamline terminology, we simply say that $f \in C(\mathbb{R}^{n \times n}_{sym})$ is *strongly symmetric quasiconvex* provided there exists $\ell > 0$ such that the function

$$(1.9) \qquad \mathbb{R}^{n \times n}_{sym} \ni z \mapsto f(z) - \ell V(z) \qquad \text{is symmetric quasiconvex,}$$

where $V := V_1$ shall be referred to as the *reduced area integrand*; the ubiquity of such functions follows along the lines of [46, Prop. 2.14] (also see ZHANG [74] for a related construction of quasiconvex, linear growth integrands). In conjunction with (LG), this condition yields the existence of BD-minima for the associated variational integrals subject to given Dirichlet data (cf. Section 6.2). Moreover, appealing to Lemma 2.7, it is closely related to *all* minimising sequences being bounded in LD(Ω). Since the 1-strong quasiconvexity for full gradient functionals, in turn, yields boundedness of the respective minimising sequences in $W^{1,1}(\Omega; \mathbb{R}^n)$ (cf. [46]), a reduction to the full gradient is systematically excluded. Within the realm of p-growth, symmetric quasiconvex functionals, this hereafter identifies the limiting case $p = 1$ as the only one requiring a treatment on its own.

In this respect, the main result of this paper reads as follows:

Theorem 1.3 (Partial regularity of BD-minimisers). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Moreover, suppose that $f: \mathbb{R}^{n \times n}_{sym} \to \mathbb{R}$ is a variational integrand that is*

(a) *of class $C^{2,1}_{loc}(\mathbb{R}^{n \times n}_{sym})$,*

(b) *of linear growth in the sense of* (LG) *and*

(c) *strongly symmetric quasiconvex in the sense of* (1.9).

Then for each $M > 0$ there exist $\varepsilon_M = \varepsilon_M(n, \ell, L, M) > 0$ and a radius $R_0 = R_0(n, \ell, L, M) > 0$ such that for every local BD-minimiser $u \in BD_{loc}(\Omega)$ of F (in the sense of (1.3)) *the following holds: If $x_0 \in \Omega$ and $0 < R < R_0$ with $B(x_0, R) \Subset \Omega$ are such that*

$$(1.10) \qquad \left| \frac{Eu(B(x_0, R))}{\mathscr{L}^n(B(x_0, R))} \right| < M \text{ and } \fint_{B(x_0, R)} \left| \mathscr{E}u - \frac{Eu(B(x_0, R))}{\mathscr{L}^n(B(x_0, R))} \right| dx + \frac{|E^s u|(B(x_0, R))}{\mathscr{L}^n(B(x_0, R))} < \varepsilon_M,$$

then u is of class $\mathrm{C}^{1,\alpha}$ on $\mathrm{B}(x_0,\frac{R}{2})$ for any $0 < \alpha < 1$. As a consequence, there exists an open subset Ω_u of Ω with $\mathscr{L}^n(\Omega \setminus \Omega_u) = 0$ such that for every $x_0 \in \Omega_u$, u is of class $\mathrm{C}^{1,\alpha}$ in a neighbourhood of x_0 for any $0 < \alpha < 1$. In particular, denoting $\Sigma_u = \Omega \setminus \Omega_u$, we have

$$\Sigma_u = \Sigma_u^1 \cup \Sigma_u^2$$

(1.11)
$$:= \left\{ x_0 \in \Omega \colon \liminf_{R \searrow 0} \left(\fint_{\mathrm{B}(x_0,R)} |\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x_0,R)}| \, \mathrm{d}x + \frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{\mathscr{L}^n(\mathrm{B}(x_0,R))} \right) > 0 \right\}$$
$$\cup \left\{ x_0 \in \Omega \colon \limsup_{R \searrow 0} \left| \frac{\mathrm{E}u(\mathrm{B}(x_0,R))}{\mathscr{L}^n(\mathrm{B}(x_0,R))} \right| = \infty \right\}.$$

Not being allowed to utilise full gradient techniques, Theorem 1.3 cannot be established in an analogous manner as its full gradient counterpart from [46]. Partially based on recently available results [11, 17, 28, 73], the proof of the previous theorem is given in Section 5. Here we rely in an essential way on an improved estimate of the local BD-minimisers' distances from suitable $\mathbb{A}$-harmonic approximations in terms of a *superlinear* power of the excess. To the best of our knowledge, an estimate of this form has only been derived recently in the BV-full gradient setup in [46], strongly relying on the full distributional gradients of BV-minima being Radon measures of finite total variation. The aforementioned superlinear excess power, in turn, stems from the higher regularity properties of the $\mathbb{A}$-harmonic approximants on *good* balls. To define the latter notion appropriately, we remark that the $\mathbb{A}$-harmonic approximants on *generic balls* receive their higher Sobolev regularity up to the boundary from the higher regularity of their prescribed Dirichlet data; the precise correspondence is displayed in Proposition 5.3. For arbitrary balls $\mathrm{B} \Subset \Omega$ and $u \in \mathrm{BD}(\Omega)$, we can only assert that $\mathrm{Tr}_{\partial \mathrm{B}}(u) \in \mathrm{L}^1(\partial \mathrm{B}; \mathbb{R}^n)$. This motivates the Fubini-type Theorem 4.1, implying that on sufficiently many spheres, BD-maps have interior traces with some additional differentiability and integrability beyond L^1. We wish to stress that by Ornstein's Non-Inequality this step does *not* follow as for BV, where the tangential traces $\partial_\tau u$ can be shown to belong to $\mathrm{L}^1(\partial \mathrm{B})$ on sufficiently many balls B (see Remark 4.2). Here we crucially use the embedding $\mathrm{BD} \hookrightarrow \mathrm{W}^{s,\frac{n}{n-1+s}}$ for $n \geq 2$, $0 < s < 1$ together with novel Poincaré-type inequalities to be proved in Section 2. Up from here, it is then the overall aim of the proof to show that Ornstein's Non-Inequality essentially becomes *invisible* throughout the comparison estimates, simultaneously keeping track of the enlarged nullspace of the symmetric gradient in comparison with that of the full gradient. This comes along with both further conceptual and technical difficulties, and Section 5 is devoted to their precise discussion and resolution. Finally, let us mention that the approach as developed here should also give a streamlining and unifying treatment for the BV-case in the dimensions $n = 2$ and $n \geq 3$; cf. Remark 5.9.

Lastly, let us comment on the hypotheses and extensions of Theorem 1.3. Condition (a) is rather of technical than instrumental nature and can be relaxed (cf. [46] for a related discussion); as our focus is on the symmetric quasiconvexity condition rather than regularity of the integrands, we stick to this assumption for simplicity. Let us note, however, that subject to (a)–(c) from above, it is moreover not too difficult to show that BD-minima are actually $\mathrm{C}^{2,\alpha}$-*partially regular once the* $\mathrm{C}^{1,\alpha}$-*regularity of Theorem 1.3 is established*. The methods underlying the proof of Theorem 1.3 also apply to suitable x-dependent integrands, whereas the case of fully non-autonomous integrands would require an additional argument. On the other hand, Theorem 1.3 exclusively establishes the partial regularity, but does not provide Hausdorff dimension bounds for the singular set Σ_u. Note that, by the strong symmetric quasiconvexity, such estimates require a refined argument; see Section 5.5 for a discussion of these matters.

1.5. **Structure of the Paper.** In Section 2, we fix notation, prove and collect miscellaneous background results. In Section 3, we deal with the superlinear growth situation and establish Theorem 1.3. Section 4 then serves to prove a Fubini-type theorem for BD that is instrumental in the proof of Theorem 1.3, and Section 5 is devoted to the proof of the latter. We conclude with an appendix in Section 6.

2. PRELIMINARIES

2.1. **General Notation.** We now briefly gather notation. Unless otherwise stated, Ω always denotes an open and bounded Lipschitz subset of $\mathbb{R}^n$. We denote $\mathrm{B}(x_0, r) := \{x \in \mathbb{R}^n \colon |x - x_0| < r\}$ the open ball of radius $r > 0$ centered at x_0 and use the symbol $\mathbb{B}^{n \times n}_{\mathrm{sym}}$ to denote the *closed unit ball* in $\mathbb{R}^{n \times n}_{\mathrm{sym}}$ with respect to the Frobenius norm $|A| := (\sum_{i,j=1}^n |a_{ij}|^2)^{1/2}$, $A = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$. Whenever X is a finite dimensional real vector space, the symbol $\langle \cdot, \cdot \rangle$ is used to denote the usual inner product on X and $\mathbb{S}(X)$ is the space of symmetric bilinear forms on X. To avoid ambiguities, note that duality pairings are exclusively used with subscripts, so e.g. $\langle \cdot, \cdot \rangle_{\mathscr{D}' \times \mathscr{D}}$ for the pairing between distributions and test functions. Also, for two given vectors $a, b \in \mathbb{R}^n$, $a \odot b := \frac{1}{2}(ab^{\mathsf{T}} + ba^{\mathsf{T}})$ denotes their symmetric tensor product, and we record that

$$(2.1) \qquad \frac{1}{\sqrt{2}} |a| \, |b| \leq |a \odot b| \leq |a| \, |b| \qquad \text{for all } a, b \in \mathbb{R}^n.$$

The symbol $\mathscr{L}(V; W)$ denotes the bounded linear operators between two normed linear spaces V and W. As usual, $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$ denote the n-dimensional Lebesgue and the $(n-1)$-dimensional Hausdorff measure, respectively, and we put $\omega_n := \mathscr{L}^n(\mathrm{B}(0,1))$. For notational brevity, we shall also sometimes write $\mathrm{d}\mathscr{H}^{n-1}(x) = \mathrm{d}\sigma_x$, but this will be clear from the context. Moreover, we denote $\mathscr{M}(\Omega; \mathbb{R}^m)$ the $\mathbb{R}^m$-valued finite Radon measures on Ω. Given $\mu \in \mathscr{M}(\Omega; \mathbb{R}^m)$ and $A \in \mathscr{B}(\Omega)$ (the Borel σ-algebra on Ω), we recall that $\mu \llcorner A := \mu(- \cap A)$ is the restriction of μ to A. When $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ or $\mu \in \mathscr{M}(\mathbb{R}^n; \mathbb{R}^m)$, we denote for a bounded set $A \in \mathscr{B}(\mathbb{R}^n)$ with $\mathscr{L}^n(A) > 0$

$$(u)_A := \fint_A u \, \mathrm{d}\mathscr{L}^n := \frac{1}{\mathscr{L}^n(A)} \int_A u \, \mathrm{d}x \ \text{ and } \ (\mu)_A := \fint_A \mathrm{d}\mu := \frac{\mu(A)}{\mathscr{L}^n(A)}.$$

If $A = \mathrm{B}(x, r)$ is a ball, we write $(u)_{x,r} := (u)_{\mathrm{B}(x,r)}$ or $(\mu)_{x,r} := (\mu)_{\mathrm{B}(x,r)}$. If, however, $A \in \mathscr{B}(\mathbb{R}^n)$ is such that $\mathscr{H}^{n-1}(A) \in (0, \infty)$ and $u \colon A \to \mathbb{R}^m$ is integrable with respect to $\mathscr{H}^{n-1}$, then we employ the notation

$$\fint_A u \, \mathrm{d}\mathscr{H}^{n-1} := \frac{1}{\mathscr{H}^{n-1}(A)} \int_A u \, \mathrm{d}\mathscr{H}^{n-1}.$$

Lastly, we denote by $c, C > 0$ generic constants that might change from line to line and shall only be specified provided their precise dependence on foregoing parameters is required. Similarly, we write $a \simeq b$ if there exist two constants $c, C > 0$ such that $ca \leq b \leq Ca$; in particular, $c, C > 0$ do not depend on a or b.

2.2. **The space** BD. In the following we recall the definition and record the properties of BD-maps as shall be required in the upcoming sections; for more detail, the reader is referred to [69, 11, 8, 4] and the references therein. We say that a measurable map $u \colon \Omega \to \mathbb{R}^n$ belongs to $\mathrm{BD}(\Omega)$ (and is then said to be of *bounded deformation*) if and only if $u \in \mathrm{L}^1(\Omega; \mathbb{R}^n)$ and

$$(2.2) \qquad |\mathrm{E}u|(\Omega) := \sup \left\{ \int_\Omega \langle u, \mathrm{div}(\varphi) \rangle \, \mathrm{d}x \colon \ \varphi \in \mathrm{C}^1_c(\Omega; \mathbb{B}^{n \times n}_{\mathrm{sym}}) \right\} < \infty.$$

The space $\mathrm{BD}_{\mathrm{loc}}(\Omega)$ then is defined in the obvious manner. Given $u \in \mathrm{BD}(\Omega)$, the Lebesgue-Radon-Nikodým decomposition of $\mathrm{E}u$ yields

$$(2.3) \qquad \mathrm{E}u = \mathrm{E}^a u + \mathrm{E}^s u = \mathscr{E}u \mathscr{L}^n \llcorner \Omega + \frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|} |\mathrm{E}^s u|,$$

where $\mathrm{E}^a u \ll \mathscr{L}^n$ and $\mathrm{E}^s u \perp \mathscr{L}^n$ are the absolutely continuous or singular parts of $\mathrm{E}u$ with respect to $\mathscr{L}^n$, respectively; in particular, we have $u \in \mathrm{LD}(\Omega)$ if and only if $u \in \mathrm{BD}(\Omega)$ and $\mathrm{E}u \ll \mathscr{L}^n$. Moreover, $\mathscr{E}u$ is the density of $\mathrm{E}^a u$ with respect to $\mathscr{L}^n$ and coincides with the symmetric part of the approximate gradient of u, cf. [4]. Throughout, we will work with the following modes of convergence: Let $u, u_1, u_2, ... \in \mathrm{BD}(\Omega)$. We say that (u_k) converges to u in the *norm topology* provided $\|u_k - u\|_{\mathrm{BD}(\Omega)} \to 0$, where $\|v\|_{\mathrm{BD}(\Omega)} := \|v\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + |\mathrm{E}v|(\Omega)$ for $v \in \mathrm{BD}(\Omega)$. On the other hand, we say that (u_k) converges to u in the[1] *weak*-sense* if $u_k \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and $\mathrm{E}u_k \overset{*}{\rightharpoonup} \mathrm{E}u$ in the sense of weak*-convergence of $\mathbb{R}^{n \times n}_{\mathrm{sym}}$-valued Radon measures on Ω, and in the *strict sense* (or *strictly*) if $u_k \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and $|\mathrm{E}u_k|(\Omega) \to |\mathrm{E}u|(\Omega)$ as $k \to \infty$. Lastly, if $u_k \to u$ strongly in $\mathrm{L}^1(\Omega;\mathbb{R}^n)$ and

$$(2.4) \qquad \int_\Omega \sqrt{1 + |\mathscr{E}u_k|^2}\,\mathrm{d}x + |\mathrm{E}^s u_k|(\Omega) \to \int_\Omega \sqrt{1 + |\mathscr{E}u|^2}\,\mathrm{d}x + |\mathrm{E}^s u|(\Omega), \qquad k \to \infty,$$

then we shall say that (u_k) converges to u in the *area-strict sense*. Note that, if we put $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$, then area-strict convergence amounts to $\langle \mathrm{E}u_k \rangle(\Omega) \to \langle \mathrm{E}u \rangle(\Omega)$ in the sense of functions of measures to be recalled in Section 2.3 below. It is then routine to show that norm implies area-strict, area-strict implies strict and strict implies weak*-convergence. When working with $u \in \mathrm{LD}(\Omega)$, we employ the norm $\|u\|_{\mathrm{LD}(\Omega)} := \|u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} + \|\varepsilon(u)\|_{\mathrm{L}^1(\Omega;\mathbb{R}^{n \times n}_{\mathrm{sym}})}$ (recall that we write $\varepsilon(u)$ for $\mathrm{E}u$ provided $\mathrm{E}u \ll \mathscr{L}^n$). Moreover, if $u \in \mathrm{BD}(\Omega)$, then there exists a sequence $(u_j) \subset \mathrm{C}^\infty(\Omega;\mathbb{R}^n) \cap \mathrm{LD}(\Omega)$ such that $u_j \to u$ strictly as $j \to \infty$; clearly, if $\Omega = \mathbb{R}^n$, we may even choose $(u_j) \subset \mathrm{C}^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$.

As is by now well-known (cf. [69, 11, 17]), Lipschitz regularity of $\partial\Omega$ implies the existence of a linear, bounded, surjective boundary trace operator $\mathrm{Tr}_{\partial\Omega} \colon \mathrm{BD}(\Omega) \to \mathrm{L}^1(\partial\Omega;\mathbb{R}^n)$, where boundedness is understood with respect to the respective norm topologies. Crucially, this operator is already surjective when acting on $\mathrm{LD}(\Omega)$. Moreover, it is also continuous for strict convergence in $\mathrm{BD}(\Omega)$ (and hence area-strict convergence, too) *but not* for weak*-convergence as specified above. Let us moreover note that, if $u \in \mathrm{BD}(\Omega)$, then the trivial extension

$$\overline{u} := \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

belongs to $\mathrm{BD}(\mathbb{R}^n)$ as well, and the operator $\mathfrak{E} \colon u \mapsto \overline{u}$ is linear and bounded from $\mathrm{BD}(\Omega)$ to $\mathrm{BD}(\mathbb{R}^n)$. We can now collect a refined result on smooth approximation, cf. [45, Sec. 4]:

Lemma 2.1 ((Area-)strict smooth approximation). *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded with Lipschitz boundary and let $u_0 \in \mathrm{LD}(\Omega)$. Then for each $u \in \mathrm{BD}(\Omega)$ there exists a sequence $(u_j) \subset u_0 + \mathrm{C}^\infty_c(\Omega;\mathbb{R}^n)$ such that $\|u_j - u\|_{\mathrm{L}^1(\Omega;\mathbb{R}^n)} \to 0$ and*

$$\int_\Omega \sqrt{1 + |\varepsilon(u_j)|^2}\,\mathrm{d}x \to \int_\Omega \sqrt{1 + |\mathscr{E}u|^2}\,\mathrm{d}x + |\mathrm{E}^s u|(\Omega) + \int_{\partial\Omega} |\mathrm{Tr}_{\partial\Omega}(u_0 - u) \odot v_{\partial\Omega}|\,\mathrm{d}\mathscr{H}^{n-1}.$$

If $\Sigma \subset \Omega$ is a C^1-manifold oriented by $v \colon \Sigma \to \mathbb{S}^{n-1}$ and $u \in \mathrm{BD}(\mathbb{R}^n)$, then $\mathrm{E}u \llcorner \Sigma$ is given by KOHN's formula (cf. [50])

$$(2.5) \qquad \mathrm{E}u \llcorner \Sigma = (u^+ - u^-) \odot v \mathscr{H}^{n-1} \llcorner \Sigma,$$

where u^+ and u^- are the right and left interior traces of u along Σ. These, in turn, are well-defined upon the orientation of v, and can be computed for $\mathscr{H}^{n-1}$-a.e. $x \in \Sigma$ by virtue of

$$(2.6) \qquad \lim_{r \searrow 0} \fint_{\Sigma^\pm(x,r)} |u(y) - u^\pm(x)|\,\mathrm{d}y = 0$$

for $\mathscr{H}^{n-1}$-a.e. $x \in \Sigma$, where $\Sigma^\pm(x,r) := \mathrm{B}(x,r) \cap \{y \in \mathbb{R}^n \colon \langle y - x, v(x) \rangle \gtrless 0\}$ for $r > 0$.

---

[1]Strictly speaking, being usually reserved for the BV-case, these notions should be termed *symmetric* weak*-, strict and area-strict convergence. As we shall work with BD exclusively, however, no confusions will arise from this.

We will also need a fractional embedding theorem for BD as one of the main ingredients in the partial regularity proof below. Let $0 < \theta < 1$ and $1 \le p < \infty$. Given $U, \Sigma \subset \mathbb{R}^n$ with $\mathscr{L}^n(U) > 0$ and $\mathscr{H}^{n-1}(\Sigma) \in (0, \infty)$, we define

$$[u]_{W^{\theta,p}(U;\mathbb{R}^m)} := \Big(\iint_{U \times U} \frac{|u(x) - u(y)|^p}{|x - y|^{n + \theta p}} \, dx \, dy \Big)^{\frac{1}{p}} \qquad \text{for } u \in L^p(U; \mathbb{R}^m),$$

$$[v]_{W^{\theta,p}(\Sigma;\mathbb{R}^m)} := \Big(\iint_{\Sigma \times \Sigma} \frac{|u(x) - u(y)|^p}{|x - y|^{n-1 + \theta p}} \, d\mathscr{H}^{n-1}(x) \, d\mathscr{H}^{n-1}(y) \Big)^{\frac{1}{p}} \quad \text{for } v \in L^p(\Sigma; \mathbb{R}^m; \mathscr{H}^{n-1}),$$

where $L^p(\Sigma; \mathbb{R}^m; \mathscr{H}^{n-1})$ is the space of maps $v \colon \Sigma \to \mathbb{R}^m$ which are p-integrable for $\mathscr{H}^{n-1}$. The full norm on $W^{\theta,p}(U; \mathbb{R}^m)$ or $W^{\theta,p}(\Sigma; \mathbb{R}^m)$ then is given by $\| \cdot \|_{W^{\theta,p}(U;\mathbb{R}^m)} := \|u\|_{L^p(U;\mathbb{R}^m)} + [u]_{W^{\theta,p}(U;\mathbb{R}^m)}$ (analogously for $W^{\theta,p}(\Sigma; \mathbb{R}^m)$). Following KOLYADA [52] (also see BOURGAIN et al. [15]), it is well-known that $BV(\Omega) \hookrightarrow W^{\theta, n/(n-1+\theta)}(\Omega)$ for $n \ge 2$ and $\theta \in (0, 1)$. For $BD(\Omega)$, we require the recent theory of VAN SCHAFTINGEN [73]: By [73, Prop. 6.3] and since $n \ge 2$ in our setting, the symmetric gradient ε is *elliptic* and *cancelling*. Writing $\varepsilon = \sum_{k=1}^{n} A_k \partial_k$ with $A_k \in \mathscr{L}(\mathbb{R}^n; \mathbb{R}^{n \times n}_{\text{sym}})$, ellipticity here means that the symbol map $\varepsilon[\xi] := \sum_{k=1}^{n} A_k \xi_k \colon \mathbb{R}^n \to \mathbb{R}^{n \times n}_{\text{sym}}$ is injective for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \setminus \{0\}$. In turn, cancellation means that the Fourier symbol $\varepsilon[\xi]$ is sufficiently spread in the sense that

$$(2.7) \qquad \bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \varepsilon[\xi](\mathbb{R}^n) = \{0\}.$$

Hence by [73, Thms. 1.3, 8.1], for each $\theta \in (0, 1)$ there exists $c = c(n, \theta) > 0$ such that[2]

$$(2.8) \quad \|\varphi\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^n)} + [\varphi]_{W^{\theta,n/(n-1+\theta)}(\mathbb{R}^n;\mathbb{R}^n)} \le c \int_{\mathbb{R}^n} |\varepsilon(\varphi)| \, dx \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n).$$

To state the next proposition, we remind the reader that on connected, open subsets Ω of $\mathbb{R}^n$, the nullspace of the symmetric gradient operator in $\mathscr{D}'(\Omega; \mathbb{R}^n)$ is given by the *rigid deformations*

$$(2.9) \qquad \mathscr{R}(\Omega) := \big\{ x \mapsto Ax + b \colon A = -A^\top, \, b \in \mathbb{R}^n \big\}.$$

If $\partial\Omega$ moreover is Lipschitz, for each $u \in BD(\Omega)$ there exists $a \in \mathscr{R}(\Omega)$ such that

$$(2.10) \qquad \|u - a\|_{L^1(\Omega;\mathbb{R}^n)} = \inf_{b \in \mathscr{R}(\Omega)} \|u - b\|_{L^1(\Omega;\mathbb{R}^n)} \le c |Eu|(\Omega),$$

where $c = c(\Omega, n) > 0$. We refer to (2.10) as the *Poincaré inequality on* $BD(\Omega)$. Now we have

Proposition 2.2. *Let $n \ge 2$ and $0 < \theta < 1$. Moreover, let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with Lipschitz boundary. Then there holds*

$$(2.11) \qquad BD(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}(\Omega; \mathbb{R}^n),$$

continuity of the embedding being understood with respect to the norm topologies.

(a) *If Ω moreover is connected, then for each $u \in BD(\Omega)$ there exists $a \in \mathscr{R}(\Omega)$ such that*

$$\|u - a\|_{W^{\theta, \frac{n}{n-1+\theta}}(\Omega;\mathbb{R}^n)} \le c |Eu|(\Omega),$$

where $c > 0$ is a constant that only depends on Ω, n and θ.

(b) *There exists a constant $c = c(n, \theta) > 0$ such that for every $x_0 \in \mathbb{R}^n$, $R > 0$ and every $u \in BD(\mathbb{R}^n)$ there exists $a \in \mathscr{R}(\mathbb{R}^n)$ such that*

$$(2.12) \qquad \Big(\fint_{B(x_0,R)} \int_{B(x_0,R)} \frac{|u_a(x) - u_a(y)|^{\frac{n}{n-1+\theta}}}{|x - y|^{n + \theta n/(n-1+\theta)}} \, dx \, dy \Big)^{\frac{n-1+\theta}{n}} \le CR^{1-\theta} \fint_{B(x_0,R)} |Eu|,$$

where $u_a := u - a$.

---

[2] Note that the embedding $BD \hookrightarrow L^{\frac{n}{n-1}}$ is originally due to STRAUSS [70].

Proof. Let $u \in \mathrm{BD}(\mathbb{R}^n)$ first and choose a sequence $(u_j) \subset \mathrm{C}_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$ such that $u_j \to u$ strictly in $\mathrm{BD}(\mathbb{R}^n)$. Passing to a suitable subsequence, we may assume without loss of generality that $u_j \to u$ $\mathscr{L}^n$-a.e. in $\mathbb{R}^n$. Therefore, by Fatou's lemma and (2.8),

$$
\begin{aligned}
\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^n)} + [u]_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\mathbb{R}^n;\mathbb{R}^n)} &\leq \liminf_{j\to\infty}\|u_j\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^n)} + [u_j]_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\mathbb{R}^n;\mathbb{R}^n)} \\
&\leq c\liminf_{j\to\infty}\int_{\mathbb{R}^n}|\varepsilon(u_j)|\,\mathrm{d}x = c\,|\mathrm{E}u|(\mathbb{R}^n),
\end{aligned}
$$
(2.13)

Now let $u \in \mathrm{BD}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and bounded with Lipschitz boundary $\partial\Omega$. By the above, there exists a bounded linear extension operator $\mathfrak{E}\colon \mathrm{BD}(\Omega) \to \mathrm{BD}(\mathbb{R}^n)$. Therefore,

$$
\begin{aligned}
\|u\|_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)} &\leq \max\{1, \mathscr{L}^n(\Omega)^{\theta/n}\}(\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n)} + [u]_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\Omega;\mathbb{R}^n)}) \\
&\leq \max\{1, \mathscr{L}^n(\Omega)^{\theta/n}\}(\|\mathfrak{E}u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n;\mathbb{R}^n)} + [\mathfrak{E}u]_{\mathrm{W}^{\theta,n/(n-1+\theta)}(\mathbb{R}^n;\mathbb{R}^n)}) \\
&\overset{(2.13)}{\leq} C(n,\theta,\mathscr{L}^n(\Omega))|\mathrm{E}\mathfrak{E}u|(\mathbb{R}^n) \leq C(n,\theta,\Omega)\|u\|_{\mathrm{BD}(\Omega)},
\end{aligned}
$$

and (2.11) follows. If Ω moreover is connected, pick $a \in \mathscr{R}(\Omega)$ such that (2.10) holds; applying the preceding inequality to $u - a$ and (2.10) consequently imply (a). Ad (b). We may assume that $x_0 = 0$, and shall write $\mathrm{B}_r := \mathrm{B}(0, r)$ for $r > 0$. Letting $u \in \mathrm{BD}(\mathbb{R}^n)$, we first determine an element $b \in \mathscr{R}(\mathbb{R}^n)$ such that $(*)$ in the following inequality holds on the unit ball, due to part (a) with $\Omega = \mathrm{B}(0, 1)$ and $p = \frac{n}{n-1+\theta}$:

$$
\begin{aligned}
\left(\int_{\mathrm{B}_R}\int_{\mathrm{B}_R}\frac{|u_b(x) - u_b(y)|^p}{|x-y|^{n+\theta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} &= \frac{R^{\frac{2n}{p}}}{R^{\frac{n+\theta p}{p}}}\left(\int_{\mathrm{B}_1}\int_{\mathrm{B}_1}\frac{|u_b(Rx) - u_b(Ry)|^p}{|x-y|^{n+\theta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}} \\
&\overset{(*)}{\leq} c\frac{R^{\frac{2n}{p}}}{R^{\frac{n+\theta p}{p}}}R\int_{\mathrm{B}_1}|(\varepsilon(u_b))(Rx)|\,\mathrm{d}x = cR^{\frac{n}{p}}R^{1-\theta}\fint_{\mathrm{B}_R}|\mathrm{E}u|.
\end{aligned}
$$

This in turn determines $a \in \mathscr{R}(\mathbb{R}^n)$ for (2.12), and the proof is hereby complete. $\qquad\square$

The dimensional hypothesis $n \geq 2$ in Proposition 2.2 in fact cannot be omitted:

Remark 2.3 ($n = 1$). The previous proposition does not remain valid for $n = 1$. This can be seen by the fact that $\mathrm{W}^{1,1}((a,b)) \not\hookrightarrow \mathrm{W}^{\theta,1/\theta}((a,b))$ for any $0 < \theta < 1$ and all $-\infty < a < b < \infty$. For example, pick $\theta = \frac{1}{2}$. Then it is well-known that $\mathrm{W}^{1,1}((a,b))$ embeds into the Besov-Nikolskiĭ-space $\mathrm{B}_{2,\infty}^{1/2}$ but not into $\mathrm{B}_{2,2}^{1/2}((a,b)) = \mathrm{W}^{1/2,2}((a,b))$. In fact, continuity of the embedding would yield that, as $n = 1$, $\mathrm{BD}((a,b)) = \mathrm{BV}((a,b))$ embeds into $\mathrm{W}^{1/2,2}((a,b))$ by smooth approximation, but the sign function belongs to $\mathrm{BV}((-1,1))$ but not to $\mathrm{W}^{1/2,2}((-1,1))$.

Remark 2.4 (Projection stability). Since $\mathscr{R}(\Omega)$ is a finite dimensional space of polynomials, the map a as in (2.10) can be taken to be the L^2-projection of u onto $\mathscr{R}(\Omega)$ (which here is well-defined for L^1-maps, too); cf. [17, Sec. 3]. In particular, it satisfies the stability estimate

$$
\fint_{\mathrm{B}(x_0,r)}|a|\,\mathrm{d}x \leq c\fint_{\mathrm{B}(x_0,r)}|u|\,\mathrm{d}x, \qquad u \in \mathrm{BD}(\mathrm{B}(x_0,r)).
$$

2.3. **Functions of measures.** Here we briefly record the most important features of functions being applied to measures. First, let $f\colon \mathbb{R}_{\mathrm{sym}}^{n\times n} \to \mathbb{R}_{\geq 0}$ be convex and satisfy the growth bound $c_1|z| - c_2 \leq f(z) \leq c_3(1 + |z|)$ for some $c_1, c_2, c_3 > 0$ and all $z \in \mathbb{R}_{\mathrm{sym}}^{n\times n}$. We recall that the *recession function* $f^\infty\colon \mathbb{R}_{\mathrm{sym}}^{n\times n} \to \mathbb{R}$ is given by

$$
f^\infty(z) := \limsup_{t\searrow 0}tf\left(\frac{z}{t}\right), \qquad z \in \mathbb{R}_{\mathrm{sym}}^{n\times n},
$$

and by convexity and the linear growth hypothesis, the lim sup is a actually a limit. Given $\mu \in \mathscr{M}(\Omega;\mathbb{R}_{\mathrm{sym}}^{n\times n})$, we denote its Lebesgue-Radon-Nikodým decomposition $\mu = \mu^a + \mu^s$ and

then define the measure $f(\mu)$ for $A \in \mathscr{B}(\Omega)$ by

$$(2.14) \qquad f(\mu)(A) := \int_A f(\mu) := \int_A f\left(\frac{\mathrm{d}\mu^a}{\mathrm{d}\mathscr{L}^n}\right)\mathrm{d}\mathscr{L}^n + \int_A f^\infty\left(\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}\right)\mathrm{d}|\mu^s|.$$

If $\xi \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$, we put $f(\mu - \xi) := f(\mu - \xi\mathscr{L}^n)$. Now suppose that $f \in \mathrm{C}(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ is merely assumed symmetric-rank-one convex (so is convex with respect to directions in the symmetric rank-one cone $\mathbb{R}^n \odot \mathbb{R}^n := \{a \odot b : \ a, b \in \mathbb{R}^n\}$) and of linear growth in the sense of (LG). Even though not giving rise to a positive measure, (2.14) still is a valid definition provided the density $\frac{\mathrm{d}\mu^s}{\mathrm{d}|\mu^s|}$ takes values in the symmetric-rank-one cone $|\mu^s|$-a.e.. In fact, in this situation, f is convex along directions contained in $\mathbb{R}^n \odot \mathbb{R}^n$ and so, by the linear growth assumption, $f^\infty(z)$ exists provided $z \in \mathbb{R}^n \odot \mathbb{R}^n$. When applying such integrands f to $\mathrm{E}u$ for $u \in \mathrm{BD}(\Omega)$, then the recent work of DE PHILIPPIS & RINDLER [28] yields $\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|} \in \mathbb{R}^n \odot \mathbb{R}^n$ $|\mathrm{E}^s u|$-a.e.. Hence

$$f(\mathrm{E}u)(A) := \int_A f(\mathrm{E}u) := \int_A f(\mathscr{E}u)\,\mathrm{d}\mathscr{L}^n + \int_A f^\infty\left(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|}\right)\mathrm{d}|\mathrm{E}^s u| \qquad \text{for all } A \in \mathscr{B}(\Omega)$$

for $u \in \mathrm{BD}(\Omega)$ is in fact a well-posed definition. Working from the previous ideas and upon the method of proof for signed variants given in [54], our fundamental background fact result is

Theorem 2.5 (RINDLER [65])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain and let $f \in \mathrm{C}(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ be a symmetric quasiconvex integrand which, in addition, satisfies* (LG)*. Also, let $u_0 \in \mathrm{BD}(\Omega)$. Then, with the notation of (2.3), the functional*

$$\overline{F}_{u_0}[u; \Omega] := \int_\Omega f(\mathscr{E}u)\,\mathrm{d}x + \int_\Omega f^\infty\left(\frac{\mathrm{d}\mathrm{E}^s u}{\mathrm{d}|\mathrm{E}^s u|}\right)\mathrm{d}|\mathrm{E}^s u| + \int_{\partial\Omega} f^\infty(\mathrm{Tr}_{\partial\Omega}(u_0 - u) \odot \nu_{\partial\Omega})\,\mathrm{d}\mathscr{H}^{n-1}$$

for $u \in \mathrm{BD}(\Omega)$ is lower semicontinuous with respect to weak*-convergence in the space $\mathrm{BD}(\Omega)$.

Finally, a lemma on the continuity of symmetric rank-one-convex functions for the area-strict metric that we shall frequently employ in conjunction with smooth approximation; in effect, it appears as a generalisation of the classical convex RESHETNYAK (semi-)continuity theory [64]:

Lemma 2.6 (Symmetric rank-one-convexity and area-strict continuity)**.** *Let $f \in \mathrm{C}(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ be symmetric rank-one convex with* (LG) *and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Then $\mathrm{BD}(\Omega) \ni u \mapsto f(\mathrm{E}u)(\Omega)$ is continuous with respect to area-strict convergence.*

The lemma follows from [17, Prop. 5.1] upon KRISTENSEN & RINDLER's refinement for signed integrands, [54, Thm. 4] and specifying to the symmetric gradient. Rather than reproducing the proof of [46, Prop. 3.1] with the relevant but easy modifications, we confine to stating the following equivalence between strong symmetric quasiconvexity at some $z_0 \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and coerciveness; recall that $\Omega \subset \mathbb{R}^n$ is assumed to be open and bounded with Lipschitz boundary.

Lemma 2.7. *Let $f \in \mathrm{C}(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ satisfy* (LG) *and let $u_0 \in \mathrm{LD}(\Omega)$ be a given Dirichlet datum. Then all minimising sequences of the variational problem*

$$(2.15) \qquad\qquad \text{to infimise } \int_\Omega f(\varepsilon(v))\,\mathrm{d}x \quad \text{over } \mathrm{LD}_{u_0}(\Omega)$$

are bounded in $\mathrm{LD}(\Omega)$ if and only if there exists $z_0 \in \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and $\ell > 0$ such that the function $h \colon z \mapsto f(z) - \ell V(z)$ is symmetric quasiconvex at z_0 (meaning that $h(z_0) \leq \int_Q h(z_0 + \varepsilon(\varphi))\,\mathrm{d}x$ for all $\varphi \in \mathrm{C}^\infty_c(Q; \mathbb{R}^n)$).

2.4. *V*-function estimates and Korn-type inequalities.

For future applications in Section 3, we record some non-standard forms of Korn-type inequalities and gather here the relevant background results from BREIT & DIENING [16]. Note that, alternatively, the specifically required forms could also be traced back to ACERBI & MINGIONE [3] but then would follow only by inspection of the proof of [3, Thm. 3.1].

A differentiable function $\psi \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ is called an *N-function* provided $\psi(0) = 0$, its derivative ψ' is right-continuous, non-decreasing and satisfies

$$(2.16) \qquad \psi'(0) = 0, \quad \psi'(t) > 0 \text{ for } t > 0 \quad \text{and} \quad \lim_{t \to \infty} \psi'(t) = \infty.$$

We now say that an *N-function* ψ is of class Δ_2 provided there exists $K > 0$ such that $\psi(2t) \leq K\psi(t)$ for all $t \geq 0$, and the infimum over all possible such constants is denoted $\Delta_2(\psi)$. Similarly, we say that an *N-function* ψ is class ∇_2 provided the Fenchel conjugate $\psi^*(t) := \sup_{s \geq 0}(st - \psi(s))$ is of class Δ_2; we put $\nabla_2(\psi) := \Delta_2(\psi^*)$. We then have

Proposition 2.8 ([16, Thm. 1.1]). *Let $\psi \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an N-function. Then the following are equivalent:*

(a) *ψ is both of class Δ_2 and ∇_2, abbreviated by $\psi \in \Delta_2 \cap \nabla_2$.*
(b) *There exists a constant $A > 0$ such that for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ there holds*

$$\int_{\mathbb{R}^n} \psi(|Du|) \, \mathrm{d}x \leq \int_{\mathbb{R}^n} \psi(A|\varepsilon(u)|) \, \mathrm{d}x.$$

(c) *There exists a constant $A' > 0$ such that for all $u \in \mathrm{C}^1(\mathbb{R}^n; \mathbb{R}^n)$ and all open balls $\mathrm{B}(x_0, R)$ there holds*

$$\int_{\mathrm{B}(x_0, R)} \psi(Du - (Du)_{\mathrm{B}(x_0, R)}) \, \mathrm{d}x \leq \int_{\mathrm{B}(x_0, R)} \psi(A'(\varepsilon(u) - (\varepsilon(u))_{\mathrm{B}(x_0, R)})) \, \mathrm{d}x.$$

Should (a) *be satisfied, then the constants from* (b) *and* (c) *only depend on $\Delta_2(\psi)$ and $\nabla_2(\psi)$.*

We next consider *shifted N-functions* (cf. [30, 31]). Letting $\varphi \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an *N-function*, we put for $a \geq 0$

$$(2.17) \qquad \varphi_a(t) := \int_0^t \frac{\varphi'(a+s)}{a+s} s \, \mathrm{d}s, \qquad t \geq 0.$$

The following lemma compactly gathers the for us most relevant results on shifted *N-functions*:

Lemma 2.9 ([30, Lem. 23]). *Let $\varphi \in \mathrm{C}^1([0, \infty)) \cap \mathrm{C}^2((0, \infty); \mathbb{R}_{\geq 0})$ be an N-function such that $c_1 t \varphi''(t) \leq \varphi'(t) \leq c_2 t \varphi''(t)$ for some $c_1, c_2 > 0$ and all $t > 0$. Given $a \geq 0$, define φ_a by* (2.17). *Then both $\Delta_2(\varphi_a)$ and $\nabla_2(\varphi_a)$ can be bounded independently of a and so φ_a satisfies the $\Delta_2 \cap \nabla_2$-condition uniformly in $a \geq 0$.*

We come to the requisite estimates for *V-functions*, which we define for $1 \leq p < \infty$ by

$$(2.18) \qquad V_p(z) := \left(1 + |z|^2\right)^{\frac{p}{2}} - 1, \qquad z \in \mathbb{R}^m,$$

so that, with the terminology of (1.9)ff., $V = V_1$; note that $V_p \in \Delta_2 \cap \nabla_2$ if and only if $1 < p < \infty$.

Lemma 2.10 ([46, Sec. 2.4], [27, Lem. 2.4]). *Let $m \in \mathbb{N}$. Then there exist constants $c > 0$ merely depending on m such that there holds*

$$(2.19) \qquad \begin{aligned} (\sqrt{2} - 1) \min\{|z|, |z|^2\} &\leq V(z) \leq \min\{|z|, |z|^2\}, \\ V(\lambda z) &\leq \lambda^2 V(z), \\ V(z + w) &\leq 2(V(z) + V(w)) \end{aligned}$$

for all $z, w \in \mathbb{R}^m$ and $\lambda \geq 1$. Moreover, for each $1 < p < \infty$ there exist two constants $0 < \theta_p \leq \Theta_p < \infty$ such that for all $z, z' \in \mathbb{R}^m$ there holds

$$(2.20) \qquad \begin{aligned} \theta_p \left(1 + |z|^2 + |z'|^2\right)^{\frac{p-2}{2}} |z'|^2 &\leq V_p(z + z') - V_p(z) - \langle V_p'(z), z' \rangle \\ &\leq \Theta_p \left(1 + |z|^2 + |z'|^2\right)^{\frac{p-2}{2}} |z'|^2. \end{aligned}$$

2.5. **Miscellaneous auxiliary results.** In this final subsection we gather some mixed technical results. We begin with the EKELAND variational principle [35], helping us to obtain good approximating sequences of certain BD-maps later on, in a form given in [42, Thm. 5.6, Rem. 5.5]:

Lemma 2.11 (Ekeland variational principle)**.** *Let* (X,d) *be a complete metric space and let* $\mathscr{F} \colon X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function for the metric topology, bounded from below and taking a finite value at some point. Assume that for some* $u \in X$ *and some* $\varepsilon > 0$ *we have* $\mathscr{F}[u] \leq \inf_X \mathscr{F} + \varepsilon$. *Then there exists* $v \in X$ *such that*

(a) $d(u,v) \leq \sqrt{\varepsilon}$,
(b) $\mathscr{F}[v] \leq \mathscr{F}[u]$,
(c) $\mathscr{F}[v] \leq \mathscr{F}[w] + \sqrt{\varepsilon} d(v,w)$ *for all* $w \in X$.

For the following, let us recall that a symmetric bilinear form $\mathbb{A} \in \mathbb{S}(\mathbb{R}^{N \times n})$ is called *strongly elliptic* or *Legendre-Hadamard* provided there exists $\lambda > 0$ such that for all $a \in \mathbb{R}^N, b \in \mathbb{R}^n$ there holds $\mathbb{A}[a \otimes b, a \otimes b] \geq \lambda |a \otimes b|^2$. For such bilinear forms, we have the following

Lemma 2.12 ([56, Lem. 15.2.1],[46, Prop. 2.11], [19, Lem. 2.11])**.** *Let* $1 < p < \infty$ *and* $k \in \mathbb{N}$. *Then, for any open ball* B *in* $\mathbb{R}^n$ *and any strongly elliptic bilinear form* $\mathbb{A} \in \mathbb{S}(\mathbb{R}^{N \times n})$, *the mapping*

$$\mathrm{W}^{k,p}(\mathrm{B};\mathbb{R}^N) \ni u \mapsto (-\operatorname{div}(\mathbb{A}\nabla u), \operatorname{Tr}_{\partial \mathrm{B}} u) \in \mathrm{W}^{k-2,p}(\mathrm{B};\mathbb{R}^N) \times \mathrm{W}^{k-\frac{1}{p},p}(\partial \mathrm{B};\mathbb{R}^N)$$

is a topological isomorphism. Moreover, if $|\mathbb{A}| \leq \Lambda$ *and* $-\operatorname{div}(\mathbb{A}\nabla u) = 0$ *in* $\mathscr{D}'(\Omega;\mathbb{R}^N)$, *then there holds* $u \in \mathrm{C}^\infty(\Omega;\mathbb{R}^N)$ *together with*

$$\sup_{\mathrm{B}(x_0,\frac{R}{2})} |\nabla u - A| + R \sup_{\mathrm{B}(x_0,\frac{R}{2})} |\nabla^2 u| \leq C \fint_{\mathrm{B}(x_0,R)} |\nabla u - A| \, \mathrm{d}x \qquad \textit{for all } A \in \mathbb{R}^{N \times n}$$

for all $\mathrm{B}(x_0,R) \Subset \Omega$, *where* $C = C(n,N,\lambda,\Lambda) > 0$ *is a constant.*

Finally, a standard iteration result:

Lemma 2.13 ([46, Lem. 4.4])**.** *Let* $\theta \in (0,1)$ *and* $R > 0$. *Suppose that* $\Phi, \Psi \colon (0,R] \to \mathbb{R}$ *are nonnegative functions such that* Φ *is bounded and* Ψ *is decreasing together with* $\Psi(\sigma t) \leq \sigma^{-2}\Psi(t)$ *for all* $t \in (0,R]$ *and* $\sigma \in (0,1]$. *Moreover, suppose that there holds*

$$\Phi(r) \leq \theta \Phi(s) + \Psi(s-r)$$

for all $r,s \in [\frac{R}{2},R]$ *with* $r < s$. *Then there exists a constant* $C = C(\theta) > 0$ *such that*

$$\Phi\left(\tfrac{R}{2}\right) \leq C\Psi(R).$$

3. THE SUPERLINEAR GROWTH CASE: PROOF OF THEOREM 1.2

In this section we provide the proof of Theorem 1.2. We thereby establish that in the superlinear growth case, the partial regularity of minima for p-strongly symmetric quasiconvex integrands, $1 < p < \infty$, can be reduced to the full gradient situation. This reduction cannot be employed in the linear growth case $p = 1$, thereby particularly motivating an independent proof of Theorem 1.3.

Henceforth, let $1 \leq p < \infty$ and suppose that $G \in \mathrm{C}(\mathbb{R}^{n \times n})$ is of p-growth in the sense that there exists $c > 0$ such that

$$(3.1) \qquad |G(z)| \leq c(1 + |z|^p)$$

for all $z \in \mathbb{R}^{n \times n}$. Recalling the function V_p from (2.18), we say that a function $G \in \mathrm{C}(\mathbb{R}^{n \times n})$ is *p-strongly quasiconvex* if and only if there exists $\lambda > 0$ such that

$$(3.2) \qquad \mathbb{R}^{n \times n} \ni z \mapsto G(z) - \lambda V_p(z) \in \mathbb{R} \qquad \text{is quasiconvex.}$$

As a consequence of the last part of Lemma 2.10, if $1 < p < \infty$, then p-strong quasiconvexity of $G \in C^2(\mathbb{R}^{n \times n})$ is equivalent to the existence of a constant $\nu > 0$ such that

$$(3.3) \qquad \nu \int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \leq \int_Q G(z + D\varphi) - G(z) \, dx$$

holds for all $z \in \mathbb{R}^{n \times n}$ and all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$. Using $(1 + t^2)^{\frac{p-2}{2}} t^2 \simeq t^2 + t^p$ for all $t \geq 0$ provided $2 \leq p < \infty$, for this range of p it is easily seen that p-strong quasiconvexity of G implies the existence of a constant $\mu > 0$ such that

$$(3.4) \qquad \mu \int_Q |D\varphi|^2 + |D\varphi|^p \, dx \leq \int_Q G(z + D\varphi) - G(z) \, dx$$

holds for all $z \in \mathbb{R}^{n \times n}$ and all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$. In particular, however, we remark that the 1-strong quasiconvexity in the sense of (3.2) is *not* equivalent to (3.3), cf. Remark 3.2. In both the superlinear and linear growth regimes, analogous statements hold for (p-)strongly symmetric quasiconvex integrands. Toward the proof of Theorem 1.2, we record the following result attributable to ACERBI & FUSCO [2] for $p \geq 2$ and CAROZZA, FUSCO and MINGIONE [19] for $1 < p < 2$:

Proposition 3.1 ([2, Thm. II.1], [19, Thm. 3.2]). *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^n$ be open. Suppose that $G \in C^2(\mathbb{R}^{n \times n})$ satisfies (3.1) for all $z \in \mathbb{R}^{n \times n}$ and,*

(a) *if $p \geq 2$, (3.4) holds for some $\mu > 0$, all $z \in \mathbb{R}^{n \times n}$ and $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$,*
(b) *if $1 < p < 2$, (3.3) holds for some $\nu > 0$, all $z \in \mathbb{R}^{n \times n}$ and $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$.*

Then for any local minimiser $u \in W_{loc}^{1,p}(\Omega; \mathbb{R}^n)$ of the integral functional

$$v \mapsto \int G(Dv) \, dx$$

there exists an open set $\Omega_u \subset \Omega$ with $\mathscr{L}^n(\Omega \setminus \Omega_u) = 0$ such that u is of class $C^{1,\alpha}$ for any $0 < \alpha < 1$ in a neighbourhood of any of the elements of Ω_u.

Working from Proposition 3.1, we proceed to establish Theorem 1.2:

Proof of Theorem 1.2, $p \geq 2$. Let $g \in C^2(\mathbb{R}_{sym}^{n \times n})$ satisfy the assumptions of Theorem 1.2. We then define a new integrand $G_g \colon \mathbb{R}^{n \times n} \to \mathbb{R}$ by

$$(3.5) \qquad G_g(z) := g(z^{sym}), \qquad z \in \mathbb{R}^{n \times n}.$$

Our aim is to establish that G_g satisfies the assumptions of Proposition 3.1. Clearly, $G_g = g \circ \Pi_{sym}$, where $\Pi_{sym} \colon \mathbb{R}^{n \times n} \to \mathbb{R}_{sym}^{n \times n}$ is the orthogonal projection onto the symmetric matrices, and hence $G_g \in C^2(\mathbb{R}^{n \times n})$. Moreover, since $|z^{sym}| \leq |z|$ for all $z \in \mathbb{R}^{n \times n}$, $|G_g(z)| \leq L_p(1 + |z^{sym}|^p) \leq L_p(1 + |z|^p)$ by (1.5), and so G_g satisfies (3.1) for all $z \in \mathbb{R}^{n \times n}$. It thus remains to show that G_g satisfies (3.4).

Recall that $g \colon \mathbb{R}_{sym}^{n \times n} \to \mathbb{R}$ satisfies (1.6). We note that, as $p \geq 2$, for each fixed $x \in Q$, the function $H_x \colon s \mapsto (1 + s^2 + |\varepsilon(u)(x)|^2)^{\frac{p-2}{2}} |\varepsilon(u)(x)|^2$ is non-decreasing and $(1 + t^2)^{\frac{p-2}{2}} t^2 \simeq t^2 + t^p$ for all $t \geq 0$. We obtain similarly to (3.3) that there exists $\widetilde{\mu} > 0$ such that

$$\widetilde{\mu} \int_Q |\varepsilon(\varphi)|^2 + |\varepsilon(\varphi)|^p \, dx \leq \int_Q g(z^{sym} + \varepsilon(\varphi)) - g(z^{sym}) \, dx$$

for all $z \in \mathbb{R}^{n \times n}$ and all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$. By the usual Korn inequalities (i.e., considering the N-function $t \mapsto t^q$ for $q > 1$ in Proposition 2.8), there exists $\mu > 0$ such that

$$\int_Q G_g(z + D\varphi) - G_g(z) \, dx = \int_Q g(z^{sym} + \varepsilon(\varphi)) - g(z^{sym}) \, dx$$

$$\geq \widetilde{\mu} \int_Q |\varepsilon(\varphi)|^2 + |\varepsilon(\varphi)|^p \, dx \geq \mu \int_Q |D\varphi|^2 + |D\varphi|^p \, dx$$

for all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$ and $z \in \mathbb{R}^{n \times n}$. Hence G_g satisfies (3.4) and so G_g satisfies the requirements of Proposition 3.1. To conclude the proof, it suffices to note that if $u \in W_{loc}^{1,p}(\Omega; \mathbb{R}^n)$ is a local minimiser for $v \mapsto \int g(\varepsilon(v)) \, dx$, then it is a local minimiser for $v \mapsto \int G_g(Dv) \, dx$. By Proposition 3.1 (a), the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Proof of Theorem 1.2, $1 < p < 2$. We may argue as in the proof in the superquadratic growth case apart from establishing the *p*-strong quasiconvexity of G_g. Note that, for $1 < p < 2$, H_x as defined above is not non-decreasing anymore; hence a more refined argument is required. To establish that G_g is *p*-strongly quasiconvex in the sense of Proposition 3.1 (b), we claim that there exists a constant $c = c(p, n) > 0$ such that

$$(3.6) \qquad \int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \le c \int_Q (1 + |z^{\mathrm{sym}}|^2 + |\varepsilon(\varphi)|^2)^{\frac{p-2}{2}} |\varepsilon(\varphi)|^2 \, dx$$

holds for all $z \in \mathbb{R}^{n \times n}$ and all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$.

In view of (3.6), let $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$ and $z \in \mathbb{R}^{n \times n}$ be arbitrary. Since $1 < p < 2$, the function $s \mapsto (1 + |s|^2 + |D\varphi(x)|^2)^{\frac{p-2}{2}}$ is decreasing in s for every $x \in Q$. Thus, as $|z^{\mathrm{sym}}| \le |z|$ for all $z \in \mathbb{R}^{n \times n}$,

$$(3.7) \qquad \int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \le \int_Q (1 + |z^{\mathrm{sym}}|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx =: (*).$$

Now, define a function $\psi \colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ by

$$(3.8) \qquad\qquad\qquad \psi(t) := (1 + t)^{p-2} t^2, \qquad t \ge 0.$$

Then we have, with the correspondingly shifted function ψ_a being defined for $a \ge 0$ by (2.17),

$$(3.9) \qquad\qquad \psi_a(t) \simeq \psi''(a + t) t^2 \simeq (1 + a + t)^{p-2} t^2 \simeq (1 + |a|^2 + |t|^2)^{\frac{p-2}{2}} t^2,$$

and the constants implicit in '$\simeq$' are *independent of a*; the lengthy yet elementary verification of this fact is deferred to the appendix, Section 6.2. As a consequence of Lemma 2.9 and $p > 1$, ψ_a belongs to $\Delta_2 \cap \nabla_2$ and, most importantly, $\Delta_2(\psi_a)$ and $\nabla_2(\psi_a)$ are independent of $a \ge 0$. Hence, by Proposition 2.8, there exists a constant $A = A(\Delta_2(\psi_a), \nabla_2(\psi_a)) > 0$ – which, since $\Delta_2(\psi_a)$ and $\nabla_2(\psi_a)$ *do not depend on a*, is actually independent of a: $A = A(\Delta_2(\psi), \nabla_2(\psi)) > 0$ – such that for all $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$ there holds

$$(3.10) \qquad\qquad\qquad \int_Q \psi_a(|D\varphi|) \, dx \le \int_Q \psi_a(A|\varepsilon(\varphi)|) \, dx.$$

Clearly, since ψ and each ψ_a are monotonically increasing, we may assume that $A > 1$. Applying the previous inequality to the particular choice $a = |z^{\mathrm{sym}}|$, we therefore obtain

$$(3.11) \qquad\qquad
\begin{aligned}
(*) &\overset{(3.9)}{\le} c \int_Q \psi_{|z^{\mathrm{sym}}|}(|D\varphi|) \, dx \\
&\overset{(3.10)}{\le} c \int_Q \psi_{|z^{\mathrm{sym}}|}(A|\varepsilon(\varphi)|) \, dx \\
&\overset{(3.9)}{\le} c \int_Q (1 + |z^{\mathrm{sym}}|^2 + A^2 |\varepsilon(\varphi)|^2)^{\frac{p-2}{2}} A^2 |\varepsilon(\varphi)|^2 \, dx \\
&\le c A^2 \int_Q (1 + |z^{\mathrm{sym}}|^2 + |\varepsilon(\varphi)|^2)^{\frac{p-2}{2}} |\varepsilon(\varphi)|^2 \, dx,
\end{aligned}$$

the last inequality being valid by $A > 1$ and $p - 2 < 0$. Then, combining (3.7) and (3.11), we arrive at (3.6). We can then proceed in showing that G_g defined by (3.5) is *p*-strongly quasiconvex. To this end, let $\varphi \in C_c^\infty(Q; \mathbb{R}^n)$ and $z \in \mathbb{R}^{n \times n}$ be arbitrary. We then find

$$\int_Q G_g(z + D\varphi) - G_g(z) \, dx = \int_Q g(z^{\mathrm{sym}} + \varepsilon(\varphi)) - g(z^{\mathrm{sym}}) \, dx$$

$$\overset{(2.20),(1.6)}{\geq} \ell_p \theta_p \int_Q (1+|z^{\mathrm{sym}}|^2 + |\varepsilon(\varphi)|^2)^{\frac{p-2}{2}} |\varepsilon(\varphi)|^2 \, \mathrm{d}x$$

$$\overset{(3.6)}{\geq} \nu \int_Q (1+|z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, \mathrm{d}x,$$

where $\nu = \frac{\ell_p \theta_p}{c}$ with $\ell_p > 0$ from (1.6) and the constant $c > 0$ from (3.6). Thus Theorem 1.2 follows for the growth range $1 < p < 2$ and in combination with the above, the proof of Theorem 1.2 is complete. $\qquad\square$

Remark 3.2. When linear growth integrands are concerned, setting $p = 1$ in (3.3) does not give rise to an equivalent notion of (1-)strong quasiconvexity in the sense of (3.2) with $p = 1$ (also see the restriction of exponents in Lemma 2.10). This can be even seen for strongly convex linear growth integrands such as the area integrand $m \colon z \mapsto \sqrt{1+|z|^2}(= V(z)+1)$, compare (5.2) from below. The underlying reason for this is that convex, linear growth C^2-integrands typically exhibit (p,q)-type growth behaviour on the level of the second derivatives in the following sense: There exist $1 < a < \infty$ and constants $\Lambda_1, \Lambda_2 > 0$ such that

$$(3.12) \qquad \Lambda_1 \frac{|z|^2}{(1+|\xi|^2)^{\frac{a}{2}}} \leq \langle m''(\xi)z, z \rangle \leq \Lambda_2 \frac{|z|^2}{(1+|\xi|^2)^{\frac{1}{2}}} \qquad \text{for all } z, \xi \in \mathbb{R}^{n \times n}_{\mathrm{sym}},$$

see [45] and [14, Ex. 4.17] for a discussion. In addition, note that if we consider an a-elliptic integrand satisfying (1.9) (the latter in turn expressing a coerciveness property on LD or BD but *not* $\mathrm{W}^{1,1}$ or BV), one obtains that generalised minima belong to $\mathrm{W}^{1,1}_{\mathrm{loc}}$ for if $1 < a < 1+\frac{2}{n}$, cf. [44, Thm. 1.1]. Even though the general reducibility to the full gradient case for $p = 1$ is ruled out by ORNSTEIN's Non-Inequality, one might hope to employ a variant of such a procedure for a-elliptic integrands, where one has improved integrability estimates. However, the available $\mathrm{W}^{1,1}_{\mathrm{loc}}$-regularity result of [44] hinges on *specific* minimising sequences being locally uniformly bounded in some $\mathrm{W}^{1,q}$, $q > 1$, but this neither implies boundedness of all minimising sequences in $\mathrm{W}^{1,1}_{\mathrm{loc}}$ nor in $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^n)$. In light of [46, Prop. 3.1], this would be necessary for a possible reduction procedure; still, if possible, it would only work for *convex* integrands with a certain ellipticity ratio, not being applicable to the strongly (symmetric) quasiconvex case.

4. A FUBINI–TYPE THEOREM FOR BD–MAPS

As one of the main tools in the proof of Theorem 1.3, we now give a Fubini-type result for functions of bounded deformation. In effect, this establishes that on $\mathscr{L}^1$-a.e. sphere with fixed center, BD-maps possess additional fractional differentiability and integrability; on arbitrary spheres, we can only expect L^1-integrability of interior traces. Aiming to linearise later on, suitable competitor maps attaining these more regular boundary values then will equally belong to better Sobolev spaces and so the results of Lemma 2.12 become accessible.

Theorem 4.1. *Let $n \geq 2$, $0 < \theta < 1$, $x_0 \in \mathbb{R}^n$ and $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$. Then for $\mathscr{L}^1$–almost all radii $r > 0$, $\mathscr{H}^{n-1}$-a.e. $x \in \partial\mathrm{B}(x_0, r)$ are Lebesgue points for u. For such $r > 0$, the restrictions $u|_{\partial\mathrm{B}(x_0,r)}$ are hereafter well-defined and moreover belong to $\mathrm{W}^{\theta, n/(n-1+\theta)}(\partial\mathrm{B}(x_0,r); \mathbb{R}^n)$.*

More precisely, there exists a constant $C = C(n, \theta) > 0$ (which, in particular, is independent of x_0 and u) with the following property: For all $0 < s < r < \infty$ there exist $t \in (s, r)$ and $\alpha \in \mathscr{R}(\mathbb{R}^n)$ such that for $\mathscr{H}^{n-1}$-a.e. $x \in \partial\mathrm{B}(x_0, t)$, $u(x)$ coincides with its precise representative and there holds

$$(4.1) \qquad \left(\fint_{\partial\mathrm{B}(x_0,t)} \int_{\partial\mathrm{B}(x_0,t)} \frac{|u_\alpha(x) - u_\alpha(y)|^{\frac{n}{n-1+\theta}}}{|x-y|^{n-1+\frac{n\theta}{n-1+\theta}}} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y \right)^{\frac{n-1+\theta}{n}} \leq C \frac{r^n}{t^{\frac{(n-1)(n-1+\theta)}{n}} (r-s)^{\frac{n-1+\theta}{n}}} \times$$
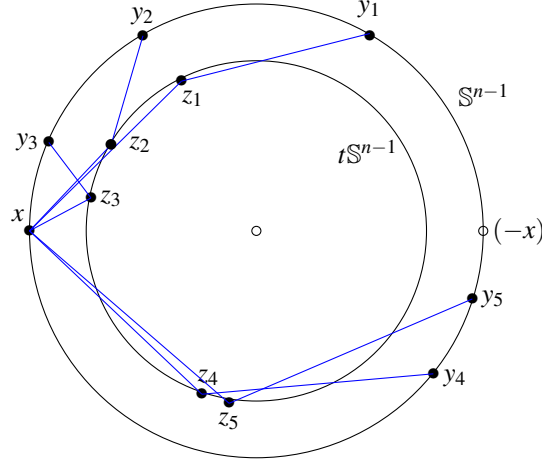
$$\times \fint_{\mathrm{B}(x_0,r)} |\mathrm{E}u|.$$

FIGURE 1. The geometric situation in the proof of Theorem 4.1 in two dimensions for selected points $y = y_i$. Excluding the $\mathscr{H}^{n-1}$–nullset $(-x)$, we project the midpoints of the line segment of x and y_i onto $t\mathbb{S}^{n-1}$. This gives rise to the projections $z_i = \pi_t(x, y_i)$, and we consequently integrate with respect to t to have the requisite second radius integral emerging.

Proof. It is no loss of generality to assume $x_0 = 0$, and hence we write $\mathrm{B}_r := \mathrm{B}(0,r)$ in the sequel. For clarity, we divide the proof into three parts.

Step 1. A general Fubini-type theorem for $\mathrm{W}^{\vartheta,p}$-maps. In a first step, we let $0 < \vartheta < 1$, $1 \leq p < \infty$ and let $u \in (\mathrm{W}^{\vartheta,p} \cap \mathrm{C})(\mathbb{R}^n; \mathbb{R}^n)$. The aim of this step is to show the inequality

$$(4.2) \qquad \int_0^R \iint_{\partial \mathrm{B}_r \times \partial \mathrm{B}_r} \frac{|u(\widetilde{x}) - u(\widetilde{y})|^p}{|\widetilde{x} - \widetilde{y}|^{n+\vartheta p - 1}} \, \mathrm{d}\sigma_{\widetilde{x}} \mathrm{d}\sigma_{\widetilde{y}} \mathrm{d}r \leq C \iint_{\mathrm{B}_R \times \mathrm{B}_R} \frac{|u(\widetilde{x}) - u(\widetilde{y})|^p}{|\widetilde{x} - \widetilde{y}|^{n+\vartheta p}} \, \mathrm{d}\widetilde{x} \mathrm{d}\widetilde{y}$$

for all $R > 0$, where $C = C(n, \vartheta, p) > 0$ is a constant. Denoting the integral on the left by $(*)$, we change variables to the unit ball and put $\widetilde{x} = rx$, $\widetilde{y} = ry$. We thereby obtain, with $\mathbb{S}^{n-1} := \partial \mathrm{B}(0,1)$,

$$(4.3) \qquad (*) \leq \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \frac{|u(rx) - u(ry)|^p}{|rx - ry|^{n+\vartheta p - 1}} \, \mathrm{d}r \mathrm{d}\sigma_y \mathrm{d}\sigma_x.$$

In comparison with the right-hand side of (4.2), the ultimate integral only contains *one* integral with respect to the radii at the cost of a lower power in the integrand's denominator. We thus must argue for the appearance of the second such integral while raising the power of the relevant integrand's denominator by 1. Let $x \in \mathbb{S}^{n-1}$ and $0 < t < R$ and be given. We put

$$\pi_t(x, y) := t \frac{x+y}{|x+y|}, \qquad y \in \mathbb{S}^{n-1} \setminus \{-x\},$$

which is the projection of the mid point of the line segment $[x,y]$ onto $\partial \mathrm{B}_t$, cf. Figure 4. Hence, the mapping $\Pi_{t,x} \colon \mathbb{S}^{n-1} \setminus \{-x\} \to t\mathbb{S}^{n-1}$ given by $\Pi_{t,x}(y) := \pi_t(x, y)$ is well–defined. We now estimate for arbitrary $x \in \mathbb{S}^{n-1}$ and $y \in \mathbb{S}^{n-1} \setminus \{-x\}$

$$|u(rx) - u(ry)|^p \leq C(|u(rx) - u(\pi_t(x,y))|^p + |u(ry) - u(\pi_t(x,y))|^p).$$

Hence for all $0 < a(x,y) \leq b(x,y) \leq R$, an integration with respect to $t \in [a(x,y), b(x,y)]$ yields

$$(4.4) \quad |u(rx) - u(ry)|^p \leq C \fint_{a(x,y)}^{b(x,y)} |u(rx) - u(\pi_t(x,y))|^p \, \mathrm{d}t + C \fint_{a(x,y)}^{b(x,y)} |u(ry) - u(\pi_t(x,y))|^p \, \mathrm{d}t.$$

At this point, fix $0 < r \leq R$. We then choose $a(x,y) := r(1 - \frac{|x-y|}{4})$ and $b(x,y) := r$. This particularly implies by $|x-y| \leq 2$ for all $x,y \in \mathbb{S}^{n-1}$

$$(4.5) \qquad |b(x,y) - a(x,y)| = r\frac{|x-y|}{4} \quad \text{and} \quad \frac{r}{2} \leq a(x,y) \leq b(x,y) = r.$$

Now, an elementary geometric consideration (cf. Figure 4) yields that for all $x \in \mathbb{S}^{n-1}$ and $y \in \mathbb{S}^{n-1} \setminus \{-x\}$ there holds

$$(4.6) \qquad \left| x - \frac{x+y}{|x+y|} \right| \leq |x-y|.$$

We thus have for all $0 < t \leq r \leq R$

$$(4.7) \qquad |rx - \pi_t(x,y)| = r\left| x - \frac{t}{r}\frac{x+y}{|x+y|} \right| \leq r\left| x - \frac{x+y}{|x+y|} \right| + r(1 - \tfrac{t}{r}) \leq r|x-y| + (r-t).$$

Combining (4.3), (4.4) and (4.5), we then obtain

$$(*) \leq C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \fint_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(\pi_t(x,y))|^p}{|rx - ry|^{n+\vartheta p - 1}} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y$$

$$+ C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \fint_{r(1-\frac{|x-y|}{4})}^r \frac{|u(ry) - u(\pi_t(x,y))|^p}{|rx - ry|^{n+\vartheta p - 1}} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_y \, \mathrm{d}\sigma_x =: \mathbf{I} + \mathbf{II},$$

where we have used that for each $x \in \mathbb{S}^{n-1}$, $\{-x\}$ is a nullset for $\mathscr{H}^{n-1}$. The two integrals are symmetric in x and y (also note that $\pi_t(x,y) = \pi_t(y,x)$), and so it suffices to employ the desired estimate for one of these two integrals. We first estimate by virtue of the first part of (4.5)

$$\mathbf{I} \leq C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(\pi_t(x,y))|^p}{|rx - ry|^{n+\vartheta p}} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y =: \mathbf{J},$$

so that the desired second radius integral has emerged. To estimate $\mathbf{J}$, note that if $r(1 - \frac{|x-y|}{4}) \leq t \leq r$, then

$$(4.8) \qquad -t \leq r\left(\frac{|x-y|}{4} - 1 \right) \Rightarrow r - t \leq r\frac{|x-y|}{4} \overset{(4.7)}{\Longrightarrow} |rx - \pi_t(x,y)| \leq \frac{5}{4}r|x-y|.$$

Moreover, we note that for such t, we have

$$(4.9) \qquad r(1 - \frac{|x-y|}{4}) \leq t \leq r \Rightarrow (1 - \frac{|x-y|}{4}) \leq \frac{t}{r} \leq 1 \Rightarrow 1 \leq \frac{r}{t} \leq \frac{1}{1 - \frac{|x-y|}{4}} \overset{|x-y|\leq 2}{\leq} 2.$$

We then estimate

$$\mathbf{J} \overset{(4.8)}{\leq} C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R (r^{n-1})^2 \int_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(\pi_t(x,y))|^p}{|rx - \pi_t(x,y)|^{n+\vartheta p}} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y$$

$$= C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(\pi_t(x,y))|^p}{|rx - \pi_t(x,y)|^{n+\vartheta p}} \left(\frac{r}{t}\right)^{n-1} t^{n-1} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y$$

$$\overset{(4.9)}{\leq} C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(\pi_t(x,y))|^p}{|rx - \pi_t(x,y)|^{n+\vartheta p}} t^{n-1} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_y = \mathbf{J}'.$$

At this point, we change variables and put $z := (x+y)/|x+y|$. By the geometry of the map $\Pi_{1,x}$ and the fact that for any $y \in \mathbb{S}^{n-1}$ there holds $\mathbb{S}^{n-1} \setminus \{-x\} \ni y \mapsto (x+y)/|x+y| \in \mathbb{S}^{n-1}$, the relevant Jacobian is seen to be bounded. A routine estimation thus yields

$$\mathbf{J}' \leq C \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \int_0^R r^{n-1} \int_{r(1-\frac{|x-y|}{4})}^r \frac{|u(rx) - u(tz)|^p}{|rx - tz|^{n+\vartheta p}} t^{n-1} \, \mathrm{d}t \, \mathrm{d}r \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_z$$

$$\leq C \int_0^R \int_0^R \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(rx) - u(tz)|^p}{|rx - tz|^{n+\vartheta p}} \, \mathrm{d}\sigma_x \, \mathrm{d}\sigma_z t^{n-1} r^{n-1} \, \mathrm{d}t \, \mathrm{d}r$$

$$\leq C \iint_{B(0,R) \times B(0,R)} \frac{|u(\widetilde{x}) - u(\widetilde{y})|^p}{|\widetilde{x} - \widetilde{y}|^{n+\vartheta p}} \, \mathrm{d}\widetilde{x} \, \mathrm{d}\widetilde{y},$$

the ultimate inequality being a direct consequence of a passage to polar coordinates; here, $C > 0$ still only depends on n, ϑ and p. This establishes (4.2) and concludes step 1.

Step 2. Existence of sufficiently many Lebesgue points. Since we finally aim to apply step 1 for the particular choice $\vartheta = \theta$ and $p = \frac{n}{n-1+\theta}$, we record that $\vartheta p < 1$ so that the traces of $\mathrm{W}^{\vartheta,p}$-maps are a priori not well-defined along $\partial \mathrm{B}_r$; thus we assumed $u \in (\mathrm{W}^{s,p} \cap \mathrm{C})(\mathbb{R}^n; \mathbb{R}^n)$ in step 1 so that this issue did not arise. In order to make use of step 1 for BD-maps u by Proposition 2.2, we start off by ensuring the explicit pointwise evaluability of u $\mathscr{H}^{n-1}$-a.e. on $\mathscr{L}^1$-a.e. sphere centered at the origin. Without loss of generality, let $u \in \mathrm{BD}(\mathbb{R}^n)$ and $0 < R_1 < R_2 < \infty$ be arbitrary. Since $|\mathrm{E}u|$ is a finite Radon measure, the set $I := \{t \in (R_1, R_2) \colon |\mathrm{E}u|(\partial \mathrm{B}_t) > 0\}$ is at most countable. Hence $\mathscr{L}^1((R_1, R_2) \setminus I) = R_2 - R_1$. Let $t \in (R_1, R_2) \setminus I$. Since $\partial \mathrm{B}_t$ is a C^1–hypersurface, (2.5) yields

$$(4.10) \qquad \mathrm{E}u \llcorner \partial \mathrm{B}_t = (u^+ - u^-) \odot \nu_{\partial \mathrm{B}_t} \, \mathscr{H}^{n-1} \llcorner \partial \mathrm{B}_t$$

with the one–sided Lebesgue limits $u^\pm$ and the outer unit normal $\nu_{\partial \mathrm{B}_t}$ to $\partial \mathrm{B}_t$. Therefore,

$$(4.11) \qquad \int_{\partial \mathrm{B}_t} |(u^+ - u^-) \odot \nu_{\partial \mathrm{B}_t}| \, \mathrm{d}\mathscr{H}^{n-1} = |\mathrm{E}u|(\partial \mathrm{B}_t) \overset{t \in (R_1, R_2) \setminus I}{=} 0.$$

This implies $|(u^+ - u^-) \odot \nu_{\partial \mathrm{B}_t}| = 0$ $\mathscr{H}^{n-1}$–a.e. on $\partial \mathrm{B}_t$, and since $|a| \, |b| \leq \sqrt{2} |a \odot b|$ by (2.1) for all $a, b \in \mathbb{R}^n$, we conclude that $\widetilde{u}(x) := u^+(x) = u^-(x)$ holds for $\mathscr{H}^{n-1}$–a.e. $x \in \partial \mathrm{B}_t$. Then, by (2.6), we have for $\mathscr{H}^{n-1}$-a.e. such $x \in \partial \mathrm{B}_t$

$$\lim_{r \searrow 0} \fint_{\mathrm{B}(x,r) \cap \{y \colon \langle y-x, \nu_{\partial \mathrm{B}_t} \rangle \geq 0\}} |u - \widetilde{u}(x)| \, \mathrm{d}\mathscr{L}^n = \lim_{r \searrow 0} \fint_{\mathrm{B}(x,r) \cap \{y \colon \langle y-x, \nu_{\partial \mathrm{B}_t} \rangle < 0\}} |u - \widetilde{u}(x)| \, \mathrm{d}\mathscr{L}^n = 0.$$

Since $\mathscr{L}^n(\mathrm{B}(x,r) \cap \{y \colon \langle y-x, \nu_{\partial \mathrm{B}_t} \rangle \gtrless 0\}) = \frac{1}{2} \mathscr{L}^n(\mathrm{B}(x,r))$, this consequently yields

$$(4.12) \qquad \lim_{r \searrow 0} \fint_{\mathrm{B}(x,r)} |u - \widetilde{u}(x)| \, \mathrm{d}\mathscr{L}^n = 0.$$

Hence, $\mathscr{H}^{n-1}$-a.e. $x \in \partial \mathrm{B}_t$ is a Lebesgue point of u. In conclusion, $\mathscr{H}^{n-1}$–a.e. $x \in \partial \mathrm{B}_t$ is a Lebesgue point for u for $\mathscr{L}^1$–a.e. radius $t \in (R_1, R_2)$.

In an intermediate step, we claim the following: Let $-\infty < a < b < \infty$ and let $J \subset (a,b)$ be a measurable subset of full Lebesgue measure, i.e., $\mathscr{L}^1((a,b) \setminus J) = 0$. Then for every $g \in \mathrm{L}^1((a,b); \mathbb{R}_{\geq 0})$ there exists $\xi_0 \in J$ which is a Lebesgue point for g and satisfies

$$(4.13) \qquad g^*(\xi_0) = \lim_{r \searrow 0} \fint_{(\xi_0 - r, \xi_0 + r)} g \, \mathrm{d}x \leq \frac{2}{b-a} \int_{(a,b)} g \, \mathrm{d}x,$$

where g^* is the precise representative of g. To see this, we note that $\mathscr{L}^1$–a.e. element of J is a Lebesgue point for g, and hence the first equality in (4.13) holds for $\mathscr{L}^1$-a.e. $\xi_0 \in J$. Assume towards a contradiction that the overall claim is wrong. Then we find $g \in \mathrm{L}^1((a,b); \mathbb{R}_{\geq 0})$ such that for all $\xi_0 \in J$ which are Lebesgue points for g there holds

$$(4.14) \qquad g^*(\xi_0) > \frac{2}{b-a} \int_{(a,b)} g(x) \, \mathrm{d}x.$$

Since this holds for $\mathscr{L}^1$–a.e. $\xi_0 \in (a,b)$, we infer by integrating with respect to $\xi_0 \in J$

$$2 \int_{(a,b)} g(x) \, \mathrm{d}x = \frac{2}{b-a} \int_{(a,b)} \int_{(a,b)} g(x) \, \mathrm{d}x \, \mathrm{d}\xi_0 \leq \int_{(a,b)} g(\xi_0) \, \mathrm{d}\xi_0.$$

By non–negativity of g, this implies $g \equiv 0$ $\mathscr{L}^1$–a.e. in (a,b). This contradicts (4.14) and the proof of the intermediate claim is complete.

Step 3. Conclusion. Let now $0 < \theta < 1$ be arbitrary and put $p := n/(n-1+\theta)$. Since the statement is local, we may assume that $u \in \mathrm{BD}_{\mathrm{loc}}(\mathbb{R}^n)$ is compactly supported. For $\varepsilon > 0$ consider the smooth approximations $u^\varepsilon(x) := \rho_\varepsilon * u(x)$, where $\rho \in \mathrm{C}_c^\infty(\mathrm{B}(0,1); [0,1])$ is a

radial mollifier with $\|\rho\|_{\mathrm{L}^1(\mathrm{B}(0,1))} = 1$, and $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$ is its ε-rescaled variant. Based on Proposition 2.2 (b), we choose a rigid deformation $\alpha_\varepsilon \in \mathscr{R}(\mathrm{B}_r)$ such that, with $u^\varepsilon_{\alpha_\varepsilon} := u^\varepsilon - \alpha_\varepsilon$,

$$(4.15) \qquad \left(\fint_{\mathrm{B}_r} \fint_{\mathrm{B}_r} \frac{|u^\varepsilon_{\alpha_\varepsilon}(x) - u^\varepsilon_{\alpha_\varepsilon}(y)|^p}{|x-y|^{n+\theta p}} \,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{p}} \le Cr^{1-\theta} \fint_{\mathrm{B}_r} |\mathrm{E}u^\varepsilon|.$$

By construction of α_ε (cf. (2.10) and Remark 2.4), $\|\alpha_\varepsilon\|_{\mathrm{L}^1(\mathrm{B}_r;\mathbb{R}^n)} \le c\|u^\varepsilon\|_{\mathrm{L}^1(\mathrm{B}_r;\mathbb{R}^n)} \to \|u\|_{\mathrm{L}^1(\mathrm{B}_r;\mathbb{R}^n)}$. Since $\mathscr{R}(\mathrm{B}_r)$ is finite dimensional, we hence find a sequence $(\varepsilon_j) \subset \mathbb{R}_{>0}$ and $\alpha \in \mathscr{R}(\mathrm{B}_r)$ such that $\alpha_{\varepsilon_j} \to \alpha$ in $\mathscr{R}(\mathrm{B}_r)$. Therefore, by Fatou's lemma and (4.2) with $R = r$ in the first step,

$$\int_s^r \iint_{\partial \mathrm{B}_t \times \partial \mathrm{B}_t} \frac{|u^*_\alpha(x) - u^*_\alpha(y)|^p}{|x-y|^{n+\theta p-1}} \,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y\,\mathrm{d}t \le C \liminf_{j\to\infty} \int_{\mathrm{B}_r} \int_{\mathrm{B}_r} \frac{|u^{\varepsilon_j}_{\alpha_{\varepsilon_j}}(x) - u^{\varepsilon_j}_{\alpha_{\varepsilon_j}}(y)|^p}{|x-y|^{n+\theta p}} \,\mathrm{d}x\,\mathrm{d}y$$

$$(4.16) \qquad\qquad\qquad\qquad \overset{(4.15)}{\le} C \liminf_{j\to\infty} r^n \left(r^{1-\theta} \fint_{\mathrm{B}_r} |\mathrm{E}u^{\varepsilon_j}| \right)^p$$

$$= Cr^n \left(r^{1-\theta} \fint_{\overline{\mathrm{B}}_r} |\mathrm{E}u| \right)^p$$

with the precise representative u^*_α of u_α. Put $J := \{t \in (s,r) \colon |\mathrm{E}u|(\partial\mathrm{B}_t) = 0\}$ and define $\lambda \colon (s,r) \to \mathbb{R}_{\ge 0}$ by

$$\lambda(t) := \iint_{\partial \mathrm{B}_t \times \partial \mathrm{B}_t} \frac{|u^*_\alpha(x) - u^*_\alpha(y)|^p}{|x-y|^{n+\theta p-1}} \,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y, \qquad t \in J$$

and $\lambda(t) = 0$ otherwise. With J from above, step 2 implies the existence of some $t \in J$ such that

$$\lambda(t) \le \frac{2}{r-s} \int_{(s,r)} \lambda(t)\,\mathrm{d}t \overset{(4.16)}{\le} C \frac{r^n}{r-s} \left(r^{1-\theta} \fint_{\overline{\mathrm{B}}_r} |\mathrm{E}u| \right)^p$$

which, upon rewriting the left-hand side of the previous inequality in terms of u^*_α, yields

$$\left(\fint_{\partial \mathrm{B}(0,t)} \int_{\partial \mathrm{B}(0,t)} \frac{|u^*_\alpha(x) - u^*_\alpha(y)|^p}{|x-y|^{n+\theta p-1}} \,\mathrm{d}\sigma_x\,\mathrm{d}\sigma_y \right)^{\frac{1}{p}} \le C \frac{r^{\frac{n}{p}} r^{1-\theta}}{t^{\frac{n-1}{p}} (r-s)^{\frac{1}{p}}} \fint_{\overline{\mathrm{B}}_r} |\mathrm{E}u|.$$

It is clear that $C > 0$ does not depend on u nor s, r, and so we arrive at (4.1); recall that $\mathscr{H}^{n-1}$-a.e. $x \in \partial\mathrm{B}_t$ is a Lebesgue point for u_α. The proof is complete. $\qquad\square$

Remark 4.2. In the BV-case, a Fubini-type property can be established by noting that for $u \in \mathrm{BV}(\mathbb{R}^n;\mathbb{R}^N)$, the tangential derivative $\partial_\tau u$ on $\mathscr{L}^1$-almost every sphere $\partial\mathrm{B}(0,t)$ is a finite Radon measure, too. This is discussed and utilised in [7] and [46]. By ORNSTEIN's Non-Inequality, we see no argument to ensure that for generic maps $u \in \mathrm{BD}(\Omega)$, $\partial_\tau u$ should be a Radon measure on even sufficiently many spheres. Also note that, by the very nature of the objects considered, any sort of 'symmetric tangential derivative' does not make sense. As to step 1 in the above proof, Fubini-type theorems for maps $u \in \mathrm{B}^s_{p,q}$ and $u \in \mathrm{F}^s_{p,q}$ have been given by TRIEBEL in the case where spheres are replaced by affine subspaces of $\mathbb{R}^n$, cf. [72, Chpt. 2.5.13]. To reduce to this setting by local coordinate transformations, transforming the left hand side of (4.2) gives rise to additional localisation terms on the right hand side. It is not clear to us how to control these to obtain the requisite form of the estimate, an issue which does not arise in the above proof.

5. THE LINEAR GROWTH CASE: PROOF OF THEOREM 1.3

This section is devoted to the proof of Theorem 1.3. Toward this objective, we aim to compare the given generalised minimiser with a suitable $\mathbb{A}$-harmonic approximations via linearisation. Since linear elliptic problems subject to L^1-boundary data are, in general, ill posed, this can only be achieved on *good* balls where the boundary traces of u share higher fractional differentiability. In this way, the corresponding $\mathbb{A}$-harmonic approximation will be well-defined; note that this unclear for *general* balls on whose boundaries a given BD-minimiser u is only known to possess traces in L^1. Consequently, this is where the Fubini-type property of BD-maps as given in

the last section enters. To arrive at the desired excess decay, we shall estimate a *V*-function-type distance of *u* to its $\mathbb{A}$-harmonic approximation in terms of a *superlinear* power of the excess, cf. Proposition 5.4. Postponing the precise discussion to Remark 5.5, a linear instead of superlinear power of the excess – which would come out by easier means – is not sufficient to conclude the excess decay. In conjunction with the Caccioppoli inequality of the second kind to be proved in Section 5.1, we will then show in Section 5.3 that the estimates gathered so far for *good* balls are in fact sufficient to conclude a preliminary excess for all relevant balls, i.e., those on which the excess does not exceed a certain constant.

In order to implement the linearisation strategy in the main part of the partial regularity proof, we introduce for $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfying (a)–(c) from Theorem 1.3 and $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ the integrands

$$(5.1) \qquad f_w(\xi) := f(\xi + w) - f(w) - \langle f'(w), \xi \rangle, \qquad \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}},$$

and remind the reader of the function $V\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ given by $V(\xi) := \sqrt{1 + |\xi|^2} - 1$.

Lemma 5.1. *For all $w, z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ we have (with an obvious interpretation for $w = 0$ or $z = 0$)*

$$(5.2) \qquad \langle V''(w)z, z \rangle = \frac{1 + |w|^2 - |w|^2 \left(\frac{w}{|w|} \cdot \frac{z}{|z|} \right)^2}{(1 + |w|^2)^{\frac{3}{2}}} \quad and \quad V_w(z) \ge \frac{1}{16} \frac{V(z)}{(1 + |w|^2)^{\frac{3}{2}}}.$$

Moreover, for each $m > 0$ there exists a constant $c = c(m) \in [1, \infty)$ with the following properties: If $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfies hypotheses (a)–(c) from Theorem 1.3, then for all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and all $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $|w| \le m$ there holds

    (i) $|f_w(z)| \le cLV(z)$,

    (ii) $|f'_w(z)| \le cL\min\{|z|, 1\}$,

    (iii) $|f''_w(0)z - f'_w(z)| \le cLV(z)$.

and for all $w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and open balls $\mathrm{B} \subset \mathbb{R}^n$ we have

$$(5.3) \qquad \frac{\ell}{c} \int_{\mathrm{B}} V(\varepsilon(\varphi)) \, \mathrm{d}x \le \int_{\mathrm{B}} f_w(\varepsilon(\varphi)) \, \mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{LD}_0(\mathrm{B}).$$

Proof. All assertions apart from (5.3) are taken from [46, Lems. 4.1, 4.2]. To see (5.3), let $\mathrm{B} \subset \mathbb{R}^n$ be an open ball and let $\varphi \in \mathrm{LD}_0(\mathrm{B}), w \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ with $|w| \le m$ be arbitrary. With condition (c) from Theorem 1.3 in the third step we deduce

$$\int_{\mathrm{B}} \frac{V(\varepsilon(\varphi)) \, \mathrm{d}x}{(1 + |w|^2)^{\frac{3}{2}}} \overset{(5.2)}{\le} 16 \int_{\mathrm{B}} V_w(\varepsilon(\varphi)) \, \mathrm{d}x = 16 \Big(\int_{\mathrm{B}} V(w + \varepsilon(\varphi)) - V(w) \, \mathrm{d}x - \underbrace{\int_{\mathrm{B}} \langle V'(w), \varepsilon(\varphi) \rangle \, \mathrm{d}x}_{=0} \Big)$$

$$\le \frac{16}{\ell} \int_{\mathrm{B}} f(w + \varepsilon(\varphi)) - f(w) \, \mathrm{d}x - \frac{16}{\ell} \underbrace{\int_{\mathrm{B}} \langle f'(w), \varepsilon(\varphi) \rangle \, \mathrm{d}x}_{=0} = \frac{16}{\ell} \int_{\mathrm{B}} f_w(\varepsilon(\varphi)) \, \mathrm{d}x.$$

Here the underbraced integrals vanish by the Gauß–Green theorem and the fact that $\varphi|_{\partial \mathrm{B}} = 0$. Noting that $|w| \le m$, (5.3) follows and the proof is complete. $\qquad\square$

5.1. **Caccioppoli inequality of the second kind.** In this section we give the requisite form of the Caccioppoli inequality of the second kind, and it is here where the BD-minimality crucially enters. However, different from other proof schemes, let us emphasize that this inequality will *not* be used to deduce higher integrability of generalised minima; in fact, GEHRING's lemma does not quite seem to fit into the linear growth framework, cf. Section 5.5 below for a discussion. From now on, we tacitly suppose that $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ satisfies (a)-(c) from Theorem 1.3 without further mentioning unless it is explicitly stated otherwise.

Proposition 5.2 (of Caccioppoli-type). *Let $m > 0$. Then there exists a constant $c = c(m, n, \frac{L}{\ell}) \in [1, \infty)$ such that if $a\colon \mathbb{R}^n \to \mathbb{R}^n$ is an affine-linear mapping with $|\mathrm{E}a| \le m$ and $\mathrm{B} = \mathrm{B}(x_0, R) \Subset \Omega$*

a ball, then there holds

$$(5.4) \qquad \int_{B(x_0, \frac{R}{2})} V(E(u-a)) \le c \int_{B(x_0,R)} V\left(\frac{u-a}{R}\right) dx$$

for every local BD*-minimiser* $u \in BD_{loc}(\Omega)$.

Proof. The proof evolves around a scheme for establishing Caccioppoli–type inequalities in the quasiconvex setting originally due to EVANS [36, Lem. 3.1]. Recalling the definition of the shifted integrands, cf. (5.1), we put $\widetilde{f} := f_{\varepsilon(a)}$ and $\widetilde{u} := u - a$. We then record that $\widetilde{u}$ is a local minimiser the functional

$$\mathscr{F}[v] := \int_\Omega \widetilde{f}(Ev)$$

over $BD_{loc}(\Omega)$. Let $\frac{R}{2} < r < s < R$ be arbitrary and choose a cut–off function $\rho \in C_c^1(B(x_0,s);[0,1])$ with $\mathbb{1}_{B(x_0,r)} \le \rho \le \mathbb{1}_{B(x_0,s)}$ and $|\nabla \rho| \le \frac{2}{s-r}$. We then define $\varphi := \rho \widetilde{u}$ and $\psi := (1-\rho)\widetilde{u}$, so that $\widetilde{u} = u - a = \varphi + \psi$. Before we continue, let us remark that with $\ell > 0$ from hypothesis (c) of Theorem 1.3 and $c = c(m) > 0$,

$$(5.5) \qquad \frac{\ell}{c} \int_{B(x_0,s)} V(E\varphi) \le \int_{B(x_0,s)} \widetilde{f}(E\varphi).$$

To see this inequality, note $\varphi|_{\partial B(x_0,s)} = 0$ and hence we find an approximating sequence $(\varphi_k) \subset C_c^\infty(B(x_0,s);\mathbb{R}^n)$ which converges in the (symmetric) area–strict sense on $B(x_0,s)$ to φ as $k \to \infty$. From Lemma 5.1, cf. (5.3), we then deduce (5.5) with φ replaced by φ_k. In the resulting inequality, by definition of (symmetric) area-strict convergence, the left-hand side converges to $\frac{\ell}{c} \int_{B(x_0,s)} V(E\varphi)$. For the right-hand side we invoke the continuity result for symmetric rank-one convex functionals with respect to symmetric area-strict convergence, cf. Lemma 2.6. By symmetric area-strict convergence and the fact that symmetric quasiconvexity implies symmetric rank-1-convexity, we hereby obtain (5.5).

Consequently, using (generalised) minimality of $\widetilde{u}$ with respect to its own boundary values and $\widetilde{u}|_{\partial B(x_0,s)} = \psi|_{\partial B(x_0,s)}$ in the second step, we obtain

$$
\begin{aligned}
\frac{\ell}{c} \int_{B(x_0,r)} V(E\widetilde{u}) \le \frac{\ell}{c} \int_{B(x_0,s)} V(E\varphi) &\le \int_{B(x_0,s)} \widetilde{f}(E\widetilde{u}) + \int_{B(x_0,s)} (\widetilde{f}(E\varphi) - \widetilde{f}(E\widetilde{u})) \qquad \text{(by (5.5))} \\
&\le \int_{B(x_0,s)} \widetilde{f}(E\psi) + \int_{B(x_0,s)} (\widetilde{f}(E\varphi) - \widetilde{f}(E\widetilde{u})) \\
&\le \int_{B(x_0,s)\setminus B(x_0,r)} \widetilde{f}(E\psi) + \int_{B(x_0,s)\setminus B(x_0,r)} (\widetilde{f}(E\varphi) - \widetilde{f}(E\widetilde{u})), \\
&=: \mathbf{I} + \mathbf{II},
\end{aligned}
$$

where the last inequality holds as $\varphi, \widetilde{u}$ coincide on $B(x_0,r)$. Then, by Lemmas 5.1(i) and 2.10,

$$
\begin{aligned}
\mathbf{I} \le cL \int_{B(x_0,s)\setminus B(x_0,r)} V(E\psi) &= cL \int_{B(x_0,s)\setminus B(x_0,r)} V\left(\left((1-\rho)\frac{dE\widetilde{u}}{d|E\widetilde{u}|}\right)|E\widetilde{u}| - (\nabla \rho \odot \widetilde{u})\mathscr{L}^n\right) \\
&\le 2cL \int_{B(x_0,s)\setminus B(x_0,r)} V(E\widetilde{u}) + 8cL \int_{B(x_0,s)} V\left(\frac{\widetilde{u}}{s-r}\right) dx
\end{aligned}
$$

On the other hand, we similarly find

$$
\begin{aligned}
\mathbf{II} &\le \int_{B(x_0,s)\setminus B(x_0,r)} \widetilde{f}\left(\left(\rho \frac{dE\widetilde{u}}{d|E\widetilde{u}|}\right)|E\widetilde{u}| + \nabla \rho \odot \widetilde{u}\mathscr{L}^n\right) - \widetilde{f}(E\widetilde{u}) \\
&\le 3cL \int_{B(x_0,s)\setminus B(x_0,r)} V(E\widetilde{u}) + 8cL \int_{B(x_0,s)} V\left(\frac{\widetilde{u}}{s-r}\right) dx.
\end{aligned}
$$

Therefore, gathering estimates, we find

$$\frac{\ell}{c} \int_{B(x_0,r)} V(E\widetilde{u}) \le 16cL \int_{B(x_0,s)\setminus B(x_0,r)} V(E\widetilde{u}) + 16cL \int_{B(x,s)} V\left(\frac{\widetilde{u}}{s-r}\right) dx.$$

We now apply WIDMAN's *hole–filling trick* and hence add $16cL\int_{B(x_0,r)}V(E\widetilde{u})$ to both sides of the previous inequality and divide the resulting inequality by $(\frac{\ell}{c}+16cL)$. In consequence, letting $\theta := 16cL/(\frac{\ell}{c}+16cL)$, we have $0 < \theta < 1$ and get

$$\int_{B(x_0,r)}V(E\widetilde{u}) \leq \theta\int_{B(x_0,s)}V(E\widetilde{u}) + \theta\int_{B(x_0,R)}V\Big(\frac{\widetilde{u}}{s-r}\Big)\mathrm{d}x.$$

From here the conclusion is immediate by Lemma 2.13. The proof is complete. $\qquad\square$

5.2. **Estimating the distance to the $\mathbb{A}$-harmonic approximation.**

In this section we present the key result that allows to deduce the requisite excess decay needed in the proof of Theorem 1.3. Here our strategy is as follows: Letting $m > 0$ be a given number and $a\colon \mathbb{R}^n \to \mathbb{R}^n$ an affine-linear map with $|Ea| \leq m$, we first establish an improved estimate for the V-function type distance of $\widetilde{u} := u - a$ to a suitable $\mathbb{A}$-harmonic approximation on *good* balls $B(x_0,R_0) \Subset \Omega$. Here *goodness* refers to balls on whose boundaries $\partial B(x_0,R_0)$ the map $\widetilde{u}$ is of class $W^{\frac{1}{n+1},\frac{n+1}{n}}(\partial B(x_0,R_0);\mathbb{R}^n)$. This is accomplished in Proposition 5.4. By the Fubini-type property of BD-maps, it is then clear that whenever $x_0 \in \Omega$ is fixed, then $\mathscr{L}^1$-a.e. radius $R_0 \in (x_0, \frac{1}{2}\operatorname{dist}(x_0,\partial\Omega))$ will qualify as a good radius. It shall then be the aim of the subsequent section to justify to have the relevant estimates on good balls to conclude a preliminary excess decay. We begin with the following proposition, making Lemma 2.12 available for the sequel.

Proposition 5.3. *Let $\mathbb{A} \in \mathbb{S}(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ be a strongly symmetric rank-one convex bilinear form, i.e., $\mathbb{A}$ satisfies for two constants $\nu_1,\nu_2 > 0$ and all $a,b \in \mathbb{R}^n$, $z_1,z_2 \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$*

$$(5.6) \qquad \nu_1|a\odot b|^2 \leq \mathbb{A}[a\odot b, a\odot b] \quad and \quad |\mathbb{A}[z_1,z_2]| \leq \nu_2|z_1||z_2|.$$

Let $Lv := -\operatorname{div}(\mathbb{A}\varepsilon(v))$, where $\mathbb{A}$ is identified with its representing matrix in $\mathbb{R}^{(n\times n)\times(n\times n)}$. Then for each $k \in \mathbb{N}$, $1 < q < \infty$ and any open ball $B \subset \mathbb{R}^n$, the mapping

$$(5.7) \qquad \Phi\colon W^{k,q}(B;\mathbb{R}^n) \ni u \mapsto (L(u), \operatorname{Tr}_{\partial B}u) \in W^{k-2,q}(B;\mathbb{R}^n) \times W^{k-\frac{1}{q},q}(\partial B;\mathbb{R}^n)$$

is a topological isomorphism. Moreover, if $u \in \mathrm{LD}(\Omega)$ satisfies $Lu = 0$ in $\mathscr{D}'(\Omega;\mathbb{R}^n)$, then there holds $u \in C^\infty(\Omega;\mathbb{R}^n)$ and

$$(5.8) \qquad \sup_{B(x_0,\frac{R}{2})}|\nabla u - A| + R\sup_{B(x_0,\frac{R}{2})}|\nabla^2 u| \leq C\fint_{B(x_0,R)}|\nabla u - A|\,\mathrm{d}x$$

for all $A \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and balls $B(x_0,R) \Subset \Omega$, where $C = C(n,\nu_1,\nu_2) > 0$ is a constant.

Proof. We reduce to Lemma 2.12 and define $\mathscr{A} \in \mathbb{S}(\mathbb{R}^{n\times n})$ by $\mathscr{A}[z_1,z_1] := \mathbb{A}[z_1^{\mathrm{sym}}, z_2^{\mathrm{sym}}]$, $z_1,z_2 \in \mathbb{R}^{n\times n}$. Then (5.6) in conjunction with (2.1) yields

$$|a\otimes b|^2 \leq |a|^2|b|^2 \leq 2|a\odot b|^2 \leq \frac{2}{\nu_1}\mathbb{A}[a\odot b, a\odot b] = \frac{2}{\nu_1}\mathscr{A}[a\otimes b, a\otimes b].$$

Hence $\mathscr{A} \in \mathbb{S}(\mathbb{R}^{n\times n})$ satisfies the hypotheses of Lemma 2.12 with $\lambda = \frac{\nu_1}{2}$. With the above terminology, we then have $\int_\Omega \mathbb{A}[\varepsilon(u), \varepsilon(\varphi)]\,\mathrm{d}x = \int_\Omega \mathscr{A}[\nabla u, \nabla\varphi]\,\mathrm{d}x$ for all $\varphi \in C^\infty_c(\Omega;\mathbb{R}^n)$. Thus, Φ given by (5.7) is a topological isomorphism by Lemma 2.12. The additional estimate (5.8) then follows similarly, now invoking the second part of Lemma 2.12. The proof is complete. $\quad\square$

We now come to the $\mathbb{A}$-harmonic approximation. Recalling that the number $m > 0$ and the affine-linear map $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $|Ea| \leq m$ are assumed fixed throughout, we put

$$\widetilde{u} := u - a.$$

Given a ball $B = B(x_0,R) \Subset \Omega$ and $u \in \mathrm{BD}(\Omega)$ with $\widetilde{u}|_{\partial B} \in W^{\frac{1}{n+1},\frac{n+1}{n}}(\partial B;\mathbb{R}^n)$, we consider the strongly symmetric rank-one system

$$(5.9) \qquad \begin{cases} -\operatorname{div}(\mathbb{A}\varepsilon(h)) = 0 & \text{in } B, \\ h = \widetilde{u} & \text{on } \partial B, \end{cases}$$

where $\mathbb{A} := \widetilde{f}''(0)$ with $\widetilde{f} := f_{\varepsilon(a)}$, cf. (5.1); note that, if f satisfies hypothesis (c) from Theorem 1.3, it is routine to check that $\mathbb{A}$ is a strongly symmetric rank-one-convex bilinear form. Put $k = 1$ and $q := 1 + \frac{1}{n}$. Then $k - \frac{1}{q} = \frac{1}{n+1}$, and in this situation Theorem 5.3 yields that there exists a unique $h \in \mathrm{W}^{1,1+1/n}(\mathrm{B}(x_0,R);\mathbb{R}^n)$ solving (5.9). We now have the following

Proposition 5.4. *Suppose that $f \in \mathrm{C}(\mathbb{R}^{n \times n}_{\mathrm{sym}})$ satisfies (a)–(c) from Theorem 1.3 and let $1 < q < \frac{n+1}{n}$, $m > 0$ be given. Then there exists a constant $C = C(m,n,q,L,\ell) > 0$ with the following property: Suppose that $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ is a local BD-minimiser for F and $\mathrm{B} = \mathrm{B}(x_0,R) \Subset \Omega$ is an open ball such that $u|_{\partial \mathrm{B}} \in \mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial \mathrm{B};\mathbb{R}^n)$. Moreover, let $a \colon \mathbb{R}^n \to \mathbb{R}^n$ be an affine–linear mapping with $|\mathrm{E}a| \le m$ and denote h the unique solution of the linear system (5.9) with $\widetilde{u} := u - a$. Then there holds*

$$(5.10) \qquad \fint_{\mathrm{B}(x_0,R)} V\Big(\frac{\widetilde{u}-h}{R}\Big)\,\mathrm{d}x \le C\Big(\fint_{\mathrm{B}(x_0,R)} V(\mathrm{E}\widetilde{u})\Big)^q.$$

Proof. We fix a ball $\mathrm{B}(x_0,R) \Subset \Omega$ such that the hypotheses of the proposition are in action. The proof then evolves in three steps:

Step 1. Ekeland approximation. To avoid manipulations on measures, we first employ an approximation procedure that allows us to work with LD- instead of BD-maps. To this end, let $\delta > 0$ be arbitrary but fixed. Then we apply the area-strict approximation of Lemma 2.1 to find $\widetilde{w}_\delta \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R)) := \widetilde{u} + \mathrm{LD}_0(\mathrm{B}(x_0,R))$ such that

$$(5.11) \qquad \begin{aligned} &\fint_{\mathrm{B}(x_0,R)} \Big|\frac{\widetilde{u}-\widetilde{w}_\delta}{R}\Big|\,\mathrm{d}x + \Big|\fint_{\mathrm{B}(x_0,R)} V(\mathrm{E}\widetilde{u}) - \fint_{\mathrm{B}(x_0,R)} V(\varepsilon(\widetilde{w}_\delta))\,\mathrm{d}x\Big| \le \delta^2, \\ &\fint_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{w}_\delta))\,\mathrm{d}x \le \fint_{\mathrm{B}(x_0,R)} \widetilde{f}(\mathrm{E}\widetilde{u}) + \delta^2, \end{aligned}$$

where the dash is understood with respect to the Lebesgue measure $\mathscr{L}^n$. Note that we can assume without loss of generality that $\widetilde{w}_\delta \in \mathrm{LD}(\mathrm{B}(x_0,R))$ since $\widetilde{u}$ only enters in the definition of $\mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$ through prescribing the traces. However, as $\mathrm{LD}(\mathrm{B}(x_0,R))$ and $\mathrm{BD}(\mathrm{B}(x_0,R))$ have the same trace space on $\partial \mathrm{B}(x_0,R)$, we can find a LD-map that has the same boundary traces on $\partial \mathrm{B}(x_0,R)$ and then proceed as before. Crucially, $(\mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R)),d_{\mathrm{sym}})$ is a complete metric space, where $d_{\mathrm{sym}}(v_1,v_2) := \|\varepsilon(v_1 - v_2)\|_{\mathrm{L}^1(\mathrm{B}(x_0,R);\mathbb{R}^{n \times n}_{\mathrm{sym}})}$ is the symmetric gradient-L^1-metric. It is then routine to check that all the requirements for the Ekeland variational principle, Lemma 2.11, are satisfied; in particular, by (6.1) from the appendix, the local BD-minimality of $\widetilde{u}$ for $v \mapsto \int \widetilde{f}(\mathrm{E}v)$ gives

$$\fint_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{w}_\delta))\,\mathrm{d}x \le \inf_{w \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))} \fint_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(w)) + \delta^2,$$

We deduce that there exists a mapping $\widetilde{v} \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$ which satisfies

$$(5.12) \qquad \begin{aligned} &\int_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{v}))\,\mathrm{d}x \le \int_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{w}_\delta))\,\mathrm{d}x, \\ &\fint_{\mathrm{B}(x_0,R)} \Big|\frac{\widetilde{v}-\widetilde{w}_\delta}{R}\Big|\,\mathrm{d}x + \fint_{\mathrm{B}(x_0,R)} |\varepsilon(\widetilde{v}) - \varepsilon(\widetilde{w}_\delta)|\,\mathrm{d}x \le (1+c_{\mathrm{Poinc}})\delta, \\ &\int_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{v}))\,\mathrm{d}x \le \int_{\mathrm{B}(x_0,R)} \widetilde{f}(\varepsilon(\widetilde{\varphi}))\,\mathrm{d}x + \delta \int_{\mathrm{B}(x_0,R)} |\varepsilon(\widetilde{v}-\widetilde{\varphi})|\,\mathrm{d}x \end{aligned}$$

for all $\widetilde{\varphi} \in \mathrm{LD}_{\widetilde{u}}(\mathrm{B}(x_0,R))$, where $c_{\mathrm{Poinc}} > 0$ is an arbitrary but fixed constant for the Poincaré inequality in $\mathrm{LD}_0(\mathrm{B}(x_0,R))$; note that the above inequality scales correctly and hence $c_{\mathrm{Poinc}} > 0$ is in fact independent of R. Working from here, we obtain

$$(5.13) \qquad \Big|\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}'(\varepsilon(\widetilde{v})), \varepsilon(\varphi)\rangle\,\mathrm{d}x\Big| \le \delta \int_{\mathrm{B}(x_0,R)} |\varepsilon(\varphi)|\,\mathrm{d}x$$

for all $\varphi \in \mathrm{LD}_0(\mathrm{B}(x_0,R))$ and

$$(5.14) \qquad \left| \int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\varepsilon(\widetilde{v}), \varepsilon(\varphi) \rangle \, \mathrm{d}x \right| \leq \int_{\mathrm{B}(x_0,R)} (cLV(\varepsilon(\widetilde{v})) + \delta) |\varepsilon(\varphi)| \, \mathrm{d}x$$

for all $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}(x_0,R);\mathbb{R}^n)$. Indeed, for every $\theta > 0$, $\widetilde{\varphi}_\theta^\pm := \widetilde{v} \pm \theta \varphi$ qualifies as a competitor in $(5.12)_3$. Hence,

$$\left| \int_{\mathrm{B}(x_0,R)} \frac{\widetilde{f}(\varepsilon(\widetilde{v})) - \widetilde{f}(\varepsilon(\widetilde{v} \pm \theta\varphi))}{\theta} \, \mathrm{d}x \right| \leq \delta \int_{\mathrm{B}(x_0,R)} |\varepsilon(\varphi)| \, \mathrm{d}x.$$

In this situation, sending $|\theta| \searrow 0$ yields (5.13). We then consequently find

$$\int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\varepsilon(\widetilde{v}), \varepsilon(\varphi) \rangle \, \mathrm{d}x \leq \int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}''(0)\varepsilon(\widetilde{v}) - \widetilde{f}'(\varepsilon(\widetilde{v})), \varepsilon(\varphi) \rangle \, \mathrm{d}x$$

$$+ \int_{\mathrm{B}(x_0,R)} \langle \widetilde{f}'(\varepsilon(\widetilde{v})), \varepsilon(\varphi) \rangle \, \mathrm{d}x \leq \int_{\mathrm{B}(x_0,R)} (cLV(\varepsilon(\widetilde{v})) + \delta) |\varepsilon(\varphi)| \, \mathrm{d}x$$

by Lemma 5.1(iii) and (5.13); note that now c depends on m. The same obviously is valid for $-\varphi$ instead of φ. This establishes (5.14). In effect, (5.13) provides perturbed Euler-Lagrange equations as a substitute for the ANZELLOTTI-type Euler-Lagrange equations for measures.

Step 2. Truncations and improved regularity for the comparison maps. Starting from (5.14), we let $\varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}(x_0,R);\mathbb{R}^n)$ be arbitrary and put $\psi := \widetilde{v} - h$. We scale back to the unit ball and therefore put, for $x \in \mathrm{B}(0,1)$,

$$\Psi(x) := \frac{1}{R} \psi(x_0 + Rx), \quad \Phi(x) := \frac{1}{R} \varphi(x_0 + Rx), \quad U(x) := \frac{1}{R} \widetilde{v}(x_0 + Rx).$$

Since h satisfies (5.9), we conclude from (5.14) with $\mathbb{A} := \widetilde{f}''(0)$

$$(5.15) \qquad \left| \int_{\mathrm{B}(0,1)} \langle \mathbb{A}\varepsilon(\Psi), \varepsilon(\Phi) \rangle \, \mathrm{d}x \right| \leq cL \int_{\mathrm{B}(0,1)} V(\varepsilon(U)) |\varepsilon(\Phi)| \, \mathrm{d}x + \delta \int_{\mathrm{B}(0,1)} |\varepsilon(\Phi)| \, \mathrm{d}x.$$

We then define a truncation operator $T \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(y) := \begin{cases} y & \text{if } |y| \leq 1, \\ \frac{y}{|y|} & \text{if } |y| > 1, \end{cases} \qquad y \in \mathbb{R}^n,$$

and note that $T(\Psi) \in \mathrm{L}^\infty(\mathrm{B}(0,1);\mathbb{R}^n)$. Let us now consider the linear system

$$(5.16) \qquad \begin{cases} -\mathrm{div}(\mathbb{A}\varepsilon(\mathbf{T})) = T(\Psi) & \text{in } \mathrm{B}(0,1), \\ \mathbf{T} = 0 & \text{on } \partial\mathrm{B}(0,1) \end{cases}$$

with its corresponding weak formulation

$$(5.17) \qquad \int_{\mathrm{B}(0,1)} \langle \mathbb{A}\varepsilon(\mathbf{T}), \varepsilon(\rho) \rangle \, \mathrm{d}x = \int_{\mathrm{B}(0,1)} \langle T(\Psi), \rho \rangle \, \mathrm{d}x \qquad \text{for all } \rho \in \mathrm{C}_c^\infty(\mathrm{B}(0,1);\mathbb{R}^n).$$

Since f is assumed strongly symmetric quasiconvex, it is strongly symmetric rank-one convex. Fix $p > n+1$. Then, by Proposition 5.3, there exists a unique solution $\mathbf{T} \in \mathrm{W}^{2,p}(\mathrm{B}(0,1);\mathbb{R}^n)$ of (5.16) with $u|_{\partial\mathrm{B}(0,1)} = 0$. Thus there exists a constant $C = C(n,p,L,\ell) > 0$ such that

$$(5.18) \qquad \int_{\mathrm{B}(0,1)} |\mathbf{T}|^p \, \mathrm{d}x + \int_{\mathrm{B}(0,1)} |\mathrm{D}\mathbf{T}|^p \, \mathrm{d}x + \int_{\mathrm{B}(0,1)} |\mathrm{D}^2\mathbf{T}|^p \, \mathrm{d}x \leq C \int_{\mathrm{B}(0,1)} |T(\Psi)|^p \, \mathrm{d}x.$$

In this situation, we invoke Morrey's embedding $\mathrm{W}^{1,p}(\mathrm{B};\mathbb{R}^n) \hookrightarrow \mathrm{C}^{0,1-n/p}(\overline{\mathrm{B}};\mathbb{R}^n)$ to find that $\mathbf{T}$ is Lipschitz together with the corresponding bound

$$(5.19) \quad \|\mathrm{D}\mathbf{T}\|_{\mathrm{L}^\infty(\mathrm{B};\mathbb{R}^{n\times n})} \leq C(\|\mathrm{D}\mathbf{T}\|_{\mathrm{L}^p(\mathrm{B};\mathbb{R}^{n\times n})} + \|\mathrm{D}^2\mathbf{T}\|_{\mathrm{L}^p(\mathrm{B};\mathbb{R}^{n\times n}\times\mathbb{R}^n)}) \overset{(5.18)}{\leq} C\|T(\Psi)\|_{\mathrm{L}^p(\mathrm{B};\mathbb{R}^n)}.$$

As $\mathbf{T}|_{\partial B(0,1)} = 0$, from here we deduce $\mathbf{T} \in W_0^{1,\infty}(B(0,1);\mathbb{R}^n)$. Approximating a generic map $\rho \in LD_0(B(0,1))$ by elements from $C_c^\infty(B(0,1);\mathbb{R}^n)$ in the LD-norm topology, we obtain

$$(5.20) \qquad \int_{B(0,1)} \langle \mathbb{A}\varepsilon(\mathbf{T}), \varepsilon(\rho) \rangle \, dx = \int_{B(0,1)} \langle T(\Psi), \rho \rangle \, dx \qquad \text{for all } \rho \in LD_0(B(0,1)).$$

Now, because of $2 \leq n < p < \infty$, we have $|T(y)|^p = |y|^p \leq |y|^2$ for if $|y| \leq 1$ and thus there holds

$$(5.21) \qquad \|T(\Psi)\|_{L^p}^p = \int_{B(0,1)} |T(\Psi)|^p \, dx \leq c \int_{B(0,1)} V(\Psi) \, dx$$

by Lemma 2.10. Combining (5.21) with (5.19) consequently yields

$$(5.22) \qquad \|\varepsilon(\mathbf{T})\|_{L^\infty} \leq \|D\mathbf{T}\|_{L^\infty} \leq c \Big(\int_{B(0,1)} V(\Psi) \, dx \Big)^{\frac{1}{p}},$$

and here $c > 0$ only depends on ℓ, L, m, n and p.

Step 3. Conclusion for the approximating maps $\widetilde{v}$. We now combine the estimates gathered so far to obtain inequality (5.10) in a perturbed form. Recalling (2.19), we succesively obtain

$$\int_{B(0,1)} V(\Psi) \, dx \overset{(2.19)_1}{\leq} \int_{B(0,1)} \min\{|\Psi|, |\Psi|^2\} \, dx$$

$$= \int_{B(0,1)} \langle T(\Psi), \Psi \rangle \, dx \qquad \text{(by definition of } T)$$

$$= \int_{B(0,1)} \langle \mathbb{A}\varepsilon(\mathbf{T}), \varepsilon(\Psi) \rangle \, dx \qquad \text{(by testing (5.20) with } \rho = \Psi)$$

$$= \int_{B(0,1)} \langle \mathbb{A}\varepsilon(\Psi), \varepsilon(\mathbf{T}) \rangle \, dx \qquad \text{(as } \mathbb{A} \in \mathbb{S}(\mathbb{R}_{sym}^{n \times n}))$$

$$\leq \int_{B(0,1)} (cLV(\varepsilon(U)) + \delta)|\varepsilon(\mathbf{T})| \, dx \qquad \text{(by testing (5.15) with } \Phi = \mathbf{T})$$

$$\leq \int_{B(0,1)} (cLV(\varepsilon(U)) + \delta) \, dx \|\varepsilon(\mathbf{T})\|_{L^\infty}$$

$$\leq \Big(\int_{B(0,1)} (cLV(\varepsilon(U)) + \delta) \, dx \Big) \Big(\int_{B(0,1)} |V(\Psi)| \, dx \Big)^{\frac{1}{p}} \qquad \text{(by (5.22)).}$$

We therefore obtain

$$(5.23) \qquad \Big(\int_{B(0,1)} V(\Psi) \, dx \Big)^{1 - \frac{1}{p}} \leq \Big(\int_{B(0,1)} (cLV(\varepsilon(U)) + \delta) \, dx \Big).$$

At this stage recall that our choice of p was only restricted to $p > n+1$. For $1 < q < \frac{n+1}{n}$ as in the proposition, we thus find $n+1 < p < \infty$ such that $p' = \frac{p}{p-1} = q$ and thus

$$(5.24) \qquad \int_{B(0,1)} V(\Psi) \, dx \leq C \Big(\int_{B(0,1)} V(\varepsilon(U)) \, dx \Big)^q + C\delta^q \mathscr{L}^n(B(0,1))^q.$$

We consequently scale back to the original ball to find

$$(5.25) \qquad \fint_{B(x_0,R)} V\Big(\frac{\widetilde{v} - h}{R} \Big) \, dx \leq C \Big(\fint_{B(0,R)} V(\varepsilon(\widetilde{v})) \, dx \Big)^q + C\delta^q \mathscr{L}^n(B(0,1))^q,$$

and we note that the constant $C > 0$ only depends on m, n, q, L and ℓ.

Step 4. Limit passage $\delta \searrow 0$ *and conclusion.* We now intend to send $\delta \searrow 0$; note that $\widetilde{v}$ actually depends on δ: $\widetilde{v} = \widetilde{v}_\delta$. By Lipschitz continuity of V we see that

$$\left| \fint_{B(x_0,R)} V\Big(\frac{\widetilde{v} - h}{R} \Big) - \fint_{B(x_0,R)} V\Big(\frac{\widetilde{u} - h}{R} \Big) \right| \leq C(V) \fint_{B(x_0,R)} \left| \frac{\widetilde{u} - \widetilde{w}_\delta}{R} \right| + C(V) \fint_{B(x_0,R)} \left| \frac{\widetilde{v} - \widetilde{w}_\delta}{R} \right|$$

$$\leq C(V)(\delta^2 + (1 + c_{\text{Poinc}})\delta) \to 0$$

by (5.11) and (5.12)$_2$ as $\delta \searrow 0$. Second, we obtain similarly

$$\left| \fint_{B(x_0,R)} V(E\widetilde{u}) - \fint_{B(x_0,R)} V(\varepsilon(\widetilde{v})) \, dx \right| \leq \left| \fint_{B(x_0,R)} V(E\widetilde{u}) - \fint_{B(x_0,R)} V(\varepsilon(\widetilde{w}_\delta)) \, dx \right|$$
$$+ C(V) \fint_{B(x_0,R)} |\varepsilon(w_\delta - \widetilde{v})| \, dx \leq (\delta^2 + C(V)\delta) \to 0$$

as $\delta \searrow 0$. In conclusion, by (5.25) we have established

$$\fint_{B(x_0,R)} V\left(\frac{\widetilde{u} - h}{R} \right) dx \leq C \left(\fint_{B(x_0,R)} V(E\widetilde{u}) \right)^q,$$

which is the desired inequality (5.10) and the proof is complete. $\qquad\square$

Remark 5.5 (On the exponent q in the previous proposition). It is important to remark that the exponent q as it appears in the previous proposition can be chosen *strictly larger* than one. From a technical perspective, the importance of $q > 1$ is given by (5.44) from below, where the smallness assumption on the excess gives smallness of the critical quantity

$$\left(\frac{\mathbf{E}(x_0, R_0)}{R_0^n} \right)^{q-1}.$$

If we could not use $q > 1$ and only had $q = 1$ at our disposal, this critical term would equal one and thus destroy the excess decay later on in Proposition 5.7.

In the preceding Proposition 5.4 we have estimated a V-function type distance of $\widetilde{u} = u - a$ to its $\mathbb{A}$-harmonic approximation h, where $\mathbb{A} = \widetilde{f}''(0) = f''_{\varepsilon(a)}(0)$. We conclude this subsection by showing how suitable Lebesgue norms of Dh can be controlled by means of $\widetilde{u}$:

Lemma 5.6. *In the situation of Proposition 5.4 there exists a constant $C = C(n, \ell, L) > 0$ such that for each $b \in \mathscr{R}(\mathbb{R}^n)$ the map $\widetilde{h} := h - b$ satisfies, with $\widetilde{u}_b := \widetilde{u} - b$,*

(5.26)
$$\left(\fint_{B(x_0,R)} |D\widetilde{h}|^{\frac{n+1}{n}} \, dx \right)^{\frac{n}{n+1}} \leq C R^{-\frac{n}{n+1}} \left(\fint_{\partial B(x_0,R)} \int_{\partial B(x_0,R)} \frac{|\widetilde{u}_b(x) - \widetilde{u}_b(y)|^{\frac{n+1}{n}}}{|x-y|^{(n-1+\frac{1}{n})}} \, d\sigma_x \, d\sigma_y \right)^{\frac{n}{n+1}}.$$

Proof. It is no loss of generality to assume $x_0 = 0$ and $R = 1$. Then we have $\widetilde{u}|_{\partial B(0,1)} \in \mathrm{W}^{\frac{1}{n}, \frac{n+1}{n}}(\partial B(0,1); \mathbb{R}^n)$. With this choice of x_0 and R, and adopting the terminology of Proposition 5.3, denote $S := \Phi^{-1}(0, \cdot)$. Given $b \in \mathscr{R}(\mathbb{R}^n)$, we define

$$\mathbf{u}_b := (\widetilde{u}_b)_{\partial B(0,1)} := \fint_{\partial B(0,1)} \widetilde{u}_b \, d\mathscr{H}^{n-1} (\in \mathbb{R}^n),$$

where the dash is now understood with respect to $\mathscr{H}^{n-1} \llcorner \partial B(0,1)$. Since h solves (5.9), $\mathbf{h} := \widetilde{h} - \mathbf{u}_b := h - b - \mathbf{u}_b$ is the unique solution of

(5.27)
$$\begin{cases} -\operatorname{div}(\mathbb{A}\varepsilon(\mathbf{h})) = 0 & \text{in } B(0,1), \\ \mathbf{h} = \widetilde{u}_b - \mathbf{u}_b & \text{on } \partial B(0,1). \end{cases}$$

Hence we have $\mathbf{h} = S(\widetilde{u}_b - \mathbf{u}_b)$ so that, by Proposition 5.3 with some $C = C(n, L, \ell) > 0$,

(5.28)
$$\|D\mathbf{h}\|_{\mathrm{L}^{\frac{n+1}{n}}(B(0,1); \mathbb{R}^{n \times n})} \leq \|\mathbf{h}\|_{\mathrm{W}^{1, \frac{n+1}{n}}(B(0,1); \mathbb{R}^n)} \leq C \|\widetilde{u}_b - \mathbf{u}_b\|_{\mathrm{W}^{\frac{1}{n+1}, \frac{n+1}{n}}(\partial B(0,1); \mathbb{R}^n)}.$$

On the other hand, if $x, y \in \partial B(0,1)$, then $|x - y| \leq 2$ and therefore $\mathscr{H}^{n-1}(\partial B(0,1))^{-1} \leq c(n)|x-y|^{-n+1-1/n}$. Thus,

$$\left(\int_{\partial B(0,1)} |\widetilde{u}_b(x) - \mathbf{u}_b|^{\frac{n+1}{n}} \, d\sigma_x \right)^{\frac{n}{n+1}} = \left(\int_{\partial B(0,1)} \fint_{\partial B(0,1)} |\widetilde{u}_b(x) - \widetilde{u}_b(y)|^{\frac{n+1}{n}} \, d\sigma_y \, d\sigma_x \right)^{\frac{n}{n+1}}$$
$$\leq C(n) \left(\int_{\partial B(x_0,1)} \int_{\partial B(0,1)} \frac{|\widetilde{u}_b(x) - \widetilde{u}_b(y)|^{\frac{n+1}{n}}}{|x-y|^{n-1+\frac{1}{n}}} \, d\sigma_y \, d\sigma_x \right)^{\frac{n}{n+1}}.$$

As a consequence, we obtain in conjunction with (5.28) and a constant $C = C(n,L,\ell) > 0$

$$(5.29) \qquad \| \mathrm{D}\widetilde{h} \|_{\mathrm{L}^{\frac{n+1}{n}}(\mathrm{B}(0,1);\mathbb{R}^{n\times n})} = \| \mathrm{D}\mathbf{h} \|_{\mathrm{L}^{\frac{n+1}{n}}(\mathrm{B}(0,1);\mathbb{R}^{n\times n})} \leq C[\widetilde{u}_b]_{\mathrm{W}^{\frac{1}{n+1},\frac{n+1}{n}}(\partial\mathrm{B}(0,1);\mathbb{R}^n)}.$$

The general case now follows by scaling, and the proof is complete. $\qquad\square$

5.3. **Excess decay.** The objective of the present subsection is to establish the excess decay that will eventually lead to the desired partial regularity assertion of Theorem 1.3 by an iteration scheme. To this end, let $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ be a local generalised minimiser of F, where the integrand satisfies (a)–(c) from Theorem 1.3, and let $M_0 > 0$ be a given number. Our strategy then runs in four steps: In a first step, we choose a ball for which both the mean value *and* a certain excess quantity of $\mathrm{E}u$ is small. Then, in a second step, we slightly diminish the radius of the given ball to obtain a ball on whose boundary we may apply the Fubini-type theorem for BD-maps. This makes the $\mathbb{A}$-harmonic approximation of the previous subsection available. Defining suitable comparison maps in step 3, we then combine Propositions 5.2 and 5.4 in step 4 to conclude a preliminary excess decay. In doing so, we define for $z \in \Omega$ and $0 < r < \mathrm{dist}(z,\partial\Omega)$ two excess quantities by

$$\mathbf{E}(u;z,r) := \int_{\mathrm{B}(z,r)} V(\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(z,r)}) \text{ and } \widetilde{\mathbf{E}}(u;z,r) := \frac{\mathbf{E}(u;z,r)}{\mathscr{L}^n(\mathrm{B}(z,r))},$$

and we will often write $\mathbf{E}(z,r) := \mathbf{E}(u;z,r)$, assuming that u is fixed. Here, as usual, $(\mathrm{E}u)_{\mathrm{B}(z,r)} = \mathrm{E}u(\mathrm{B}(z,r))/\mathscr{L}^n(\mathrm{B}(z,r))$.

Step 1. Smallness Assumptions. Let $M_0 > 0$ be given and fix a ball $\mathrm{B}_{R_0} = \mathrm{B}(x_0,R_0) \Subset \Omega$ such that

$$(5.30) \qquad\qquad |(\mathrm{E}u)_{\mathrm{B}_{R_0}}| \leq M_0.$$

and

$$(5.31) \qquad\qquad \fint_{\mathrm{B}_{R_0}} |\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}_{R_0}}| \leq 1.$$

We write $\mathrm{B}_r := \mathrm{B}(x_0,r)$ in all of what follows.

Step 2. Selection of a good radius. In a second step, we fix an affine–linear map $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $\varepsilon(a) = (\mathrm{E}u)_{\mathrm{B}_{R_0}}$. We then put $\widetilde{u} := u - a$ and $\widetilde{f} := f_{\varepsilon(a)}$, cf. (5.1). Starting from $R_0 > 0$ as given above, we now apply Theorem 4.1. Consequently, we find $R \in \left(\frac{9}{10}R_0, \frac{19}{20}R_0\right)$ such that $\widetilde{u}|_{\partial\mathrm{B}_R} \in \mathrm{W}^{\frac{1}{n+1},1+\frac{1}{n}}(\partial\mathrm{B}_R;\mathbb{R}^n)$ and a rigid deformation $b \in \mathscr{R}(\mathbb{R}^n)$ together with the corresponding estimate (with $\theta = \frac{1}{n+1}$ and accordingly $p = \frac{n+1}{n}$ in Theorem 4.1)

$$\left(\fint_{\partial\mathrm{B}_R} \int_{\partial\mathrm{B}_R} \frac{|\widetilde{u}_b(x) - \widetilde{u}_b(y)|^{1+\frac{1}{n}}}{|x-y|^{(n-1)+\frac{1}{n}}} \, \mathrm{d}\sigma_x \mathrm{d}\sigma_y \right)^{\frac{n}{n+1}}$$

$$(5.32) \qquad\qquad \leq C \frac{\left(\frac{19}{20}R_0\right)^n}{\left(\frac{9}{10}R_0\right)^{\frac{n(n-1)}{n+1}} \left(\frac{1}{20}R_0\right)^{\frac{n}{n+1}}} \frac{1}{\left(\frac{19}{20}R_0\right)^n} \int_{\mathrm{B}_{19R_0/20}} |\mathrm{E}\widetilde{u}|$$

$$\leq C \frac{R_0^{\frac{n}{n+1}}}{R_0^n} \int_{\mathrm{B}_{R_0}} |\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}_{R_0}}|.$$

where we recall $\widetilde{u}_b := \widetilde{u} - b (= u - a - b)$, and $C = C(n) > 0$ is a constant.

Step 3. Definition of comparison maps. We put $\mathbb{A} := f''((\mathrm{E}u)_{\mathrm{B}_R})$ and pick the $\mathbb{A}$–harmonic mapping $\widetilde{h}\colon \mathrm{B}_R \to \mathbb{R}^n$ solving

$$(5.33) \qquad\qquad \begin{cases} -\mathrm{div}(f''((\mathrm{E}u)_{\mathrm{B}_R})\varepsilon(\widetilde{h})) = 0 & \text{in } \mathrm{B}_R, \\ \widetilde{h} = \widetilde{u}_b & \text{on } \partial\mathrm{B}_R. \end{cases}$$

We are thus in the setting of (5.9) and Lemma 5.6 from above; by Proposition 5.3, $\widetilde{h} \in C^\infty(B_R; \mathbb{R}^n)$. Then we define

$$(5.34) \qquad A(x) := \widetilde{h}(x_0) + D\widetilde{h}(x_0)(x - x_0) \quad \text{and} \quad a_0(x) := a(x) + A(x), \qquad x \in B_R.$$

We then obtain

$$\begin{aligned}
|\varepsilon(a_0)| = |\varepsilon(a) + \varepsilon(\widetilde{h})(x_0)| &= |(Eu)_{B_{R_0}} + \varepsilon(\widetilde{h})(x_0)| \\
&\leq M_0 + |\varepsilon(\widetilde{h})(x_0)| && \text{(by (5.30))} \\
&\leq M_0 + \sup_{B_{R/2}} |D\widetilde{h}| && \text{(as } |Ev| \leq |Dv|) \\
&\leq M_0 + c \fint_{B_R} |D\widetilde{h}| \, dx && \text{(by (5.8))} \\
&\leq M_0 + c \Big(\fint_{B_R} |D\widetilde{h}|^{\frac{n+1}{n}} \, dx \Big)^{\frac{n}{n+1}}
\end{aligned}$$

and thus

$$(5.35) \qquad \begin{aligned}
|\varepsilon(a_0)| &\leq M_0 \\
&\quad + cR^{-\frac{n}{n+1}} \Big(\fint_{\partial B_R} \int_{\partial B_R} \frac{|\widetilde{u}_b(x) - \widetilde{u}_b(y)|^{\frac{n+1}{n}}}{|x-y|^{n-1+\frac{1}{n}}} \, d\sigma_x \, d\sigma_y \Big)^{\frac{n}{n+1}} && \text{(by Lemma 5.6)} \\
&\leq M_0 + \frac{c}{R_0^n} \int_{B(x_0,R_0)} |Eu - (Eu)_{B_{R_0}}| && \text{(by (5.32) and } R_0 \sim R) \\
&\leq M_0 + c,
\end{aligned}$$

where the last estimates holds because of (5.31). Here, $c = c(n, L, \ell) > 0$ is a constant that we fix now. In particular, the constants appearing here do not depend on R or R_0. Summarising, if we put $m := M_0 + c$ as on the right side of the previous chain of inequalities, then we obtain

$$(5.36) \qquad |\varepsilon(a_0)| \leq m.$$

Step 4. Comparison estimates. Let $0 < \sigma < \frac{1}{5}$ be arbitrary. We note, as a consequence of Lemma 2.10 and Jensen's inequality,

$$(5.37) \qquad \begin{aligned}
\int_{B_{\sigma R_0}} V(Eu - (Eu)_{B_{\sigma R_0}}) &= \int_{B_{\sigma R_0}} V(Eu - Ea_0 + Ea_0 - (Eu)_{B_{\sigma R_0}}) \\
&\leq C \int_{B_{\sigma R_0}} V(Eu - Ea_0) + C \int_{B_{\sigma R_0}} V((E(u - a_0))_{B_{\sigma R_0}}) \\
&\leq C \int_{B_{\sigma R_0}} V(E(u - a_0)) \overset{b \in \mathscr{R}(\mathbb{R}^n)}{=} C \int_{B_{\sigma R_0}} V(E(u - b - a_0)).
\end{aligned}$$

Our next objective is to apply the Caccioppoli–type inequality, Proposition 5.2. Having chosen $m > 0$ as it appears in Proposition 5.2 by (5.36), we find $c = c(m, n, L, \ell) > 0$ such that (5.4) holds with the requisite modifications; note that $b + a_0$ is affine-linear, too, with $|\varepsilon(b + a_0)| = |\varepsilon(a_0)| \leq m$. We then estimate, using (5.37) and the Caccioppoli–type inequality in the first step,

$$\begin{aligned}
\int_{B_{\sigma R_0}} V(Eu - (Eu)_{B_{\sigma R_0}}) &\leq C \int_{B_{2\sigma R_0}} V\Big(\frac{\widetilde{u}(x) - b(x) - A(x)}{\sigma R_0} \Big) \, dx \\
&\leq C \int_{B_{2\sigma R_0}} V\Big(\frac{\widetilde{u}(x) - b(x) - \widetilde{h}(x) - A(x) + \widetilde{h}(x)}{\sigma R_0} \Big) \, dx \\
&\leq C \int_{B_{2\sigma R_0}} V\Big(\frac{(\widetilde{u} - b) - \widetilde{h}}{\sigma R_0} \Big) \, dx + C \int_{B_{2\sigma R_0}} V\Big(\frac{\widetilde{h} - A}{\sigma R_0} \Big) \, dx
\end{aligned}$$

$$\leq \frac{C}{\sigma^2} \int_{B_R} V\Big(\frac{\widetilde{u}_b - \widetilde{h}}{R}\Big)\,\mathrm{d}x + C \int_{B_{2\sigma R_0}} V\Big(\frac{\widetilde{h} - A}{\sigma R_0}\Big)\,\mathrm{d}x$$

$$=: \mathbf{I} + \mathbf{II},$$

where $C = C(m, n, \frac{L}{\ell}) > 0$ is a constant. Here we have used $B_{2\sigma R_0} \subset B_R$, uniform comparability of R and R_0 and the fact that $V(\lambda z) \leq c\lambda^2 V(z)$ for a constant $c > 0$, all $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ and $|\lambda| \geq 1$ (cf. Lemma 2.10). We continue with the estimation of $\mathbf{I}$, and for this purpose let $1 < q < \frac{n+1}{n}$ be arbitrary but fixed. We use Proposition 5.4 and uniform comparability of R, R_0 to obtain

$$(5.38) \qquad \mathbf{I} = \frac{C}{\sigma^2} \int_{B_R} V\Big(\frac{\widetilde{u}_b - \widetilde{h}}{R}\Big)\,\mathrm{d}x = \frac{CR^n}{\sigma^2} \fint_{B_R} V\Big(\frac{\widetilde{u} - h}{R}\Big)\,\mathrm{d}x \leq C\frac{R_0^n}{\sigma^2}\Big(\fint_{B_R} V(\mathrm{E}\widetilde{u})\Big)^q,$$

the last step being valid by uniform comparability of R and R_0. As usual, the map h is defined as the solution of the strongly symmetric elliptic system (5.33) with boundary datum $\widetilde{u} = u - a$. As to $\mathbf{II}$, let $x \in B_{2\sigma R_0}$. We employ a pointwise estimate to find by use of Taylor's formula

$$\left|\frac{\widetilde{h}(x) - A(x)}{\sigma R_0}\right| = \left|\frac{\widetilde{h}(x) - \widetilde{h}(x_0) - \langle \mathrm{D}\widetilde{h}(x_0), x - x_0\rangle}{\sigma R_0}\right|$$

$$\leq C\Big(\sup_{B_{R/2}}|\mathrm{D}^2\widetilde{h}|\Big)\frac{|x - x_0|^2}{\sigma R_0}$$

$$\leq C\Big(\sup_{B_{R/2}}|\mathrm{D}^2\widetilde{h}|\Big)\frac{(2\sigma R_0)^2}{\sigma R_0} \qquad\qquad (\text{since } x \in B(x_0, 2\sigma R_0))$$

$$\leq C\sigma R\Big(\sup_{B_{R/2}}|\mathrm{D}^2\widetilde{h}|\Big) \qquad\qquad (\text{since } R_0 \leq \tfrac{10}{9}R)$$

$$\leq C\sigma \fint_{B_R}|\mathrm{D}\widetilde{h}|\,\mathrm{d}x \qquad\qquad (\text{by Proposition 5.3})$$

$$\leq C\sigma\Big(\fint_{B_R}|\mathrm{D}\widetilde{h}|^{\frac{n+1}{n}}\,\mathrm{d}x\Big)^{\frac{n}{n+1}} =: \mathbf{III} \qquad\qquad (\text{by Jensen}).$$

Similarly as in the estimation given in (5.35), we again employ Lemma 5.6 to further obtain

$$\mathbf{III} \leq C\sigma \fint_{B_{R_0}} |\mathrm{E}\widetilde{u}| \overset{\mathrm{Def}}{=} C\sigma \fint_{B_{R_0}} |\mathrm{E}(u - a)| \qquad\qquad (\text{by Lemma 5.6 and (5.32)})$$

$$= C\sigma \fint_{B_{R_0}} |\mathrm{E}u - (\mathrm{E}u)_{B_{R_0}}| \qquad\qquad (\text{since } \mathrm{E}a = (\mathrm{E}u)_{B_{R_0}})$$

$$= C\sigma\Big(\fint_{B_{R_0}} |\mathrm{E}u - (\mathrm{E}u)_{B_{R_0}}|\Big)^{2\cdot\frac{1}{2}}$$

$$\leq C\sigma\Big(V\Big(\fint_{B_{R_0}} |\mathrm{E}u - (\mathrm{E}u)_{B_{R_0}}|\Big)\Big)^{\frac{1}{2}} \qquad\qquad (\text{by (5.31) and (2.19)}_1)$$

$$\leq C\sigma\Big(\fint_{B_{R_0}} V(|\mathrm{E}u - (\mathrm{E}u)_{B_{R_0}}|)\Big)^{\frac{1}{2}} \qquad\qquad (\text{by Jensen and } \tfrac{9}{10}R_0 < R < R_0).$$

Collecting estimates, we obtain with a constant $C = C(m, n, L, \ell) > 0$ and for all $x \in B_{2\sigma R_0}$

$$(5.39) \qquad V\Big(\frac{\widetilde{h}(x) - A(x)}{\sigma R_0}\Big) \leq CV\Big(\sigma\Big(\fint_{B_{R_0}} V(|\mathrm{E}u - (\mathrm{E}u)_{B_{R_0}}|)\Big)^{\frac{1}{2}}\Big) =: CV(\Upsilon),$$

where Υ is defined in the obvious manner. Now, since $V(\cdot) \le |\cdot|$, $0 < \sigma < \frac{1}{5}$ and by (5.31), $\Upsilon \le 1$. Consequently, integrating (5.39) over $\mathrm{B}_{2\sigma R_0}$, we obtain with $C = C(m,n,L,\ell) > 0$

$$
\mathbf{II} = C \int_{\mathrm{B}_{2\sigma R_0}} V\left(\frac{\widetilde{h}(x) - A(x)}{\sigma R_0} \right) \mathrm{d}x \le C(\sigma R_0)^n V(\Upsilon)
$$

(5.40)

$$
\overset{(2.19)_1}{\le} C(\sigma R_0)^n \min\{\Upsilon, \Upsilon^2\} \le C\sigma^{n+2} R_0^n \fint_{\mathrm{B}_{R_0}} V(|\mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}_{R_0}}|).
$$

Combining estimates (5.38) and (5.40), we then find with a constant $C = C(n,m,L,\ell,q) > 0$ that

$$
\mathbf{E}(x_0,\sigma R_0) \le C \frac{R_0^n}{\sigma^2} \left(\fint_{\mathrm{B}_R} V(\mathrm{E}\widetilde{u}) \right)^q + C\sigma^{n+2} R_0^n \fint_{\mathrm{B}_R} V(|\mathrm{E}\widetilde{u}|)
$$

(5.41)

$$
\le \frac{C}{\sigma^2} \left(\frac{\mathbf{E}(x_0,R_0)}{R_0^n} \right)^{q-1} \mathbf{E}(x_0,R_0) + C\sigma^{n+2} \mathbf{E}(x_0,R_0)
$$

$$
= \left(\frac{C}{\sigma^2} (\widetilde{\mathbf{E}}(x_0,R_0))^{q-1} + C\sigma^{n+2} \right) \mathbf{E}(x_0,R_0).
$$

We will now use the previous inequality to deduce a preliminary excess decay.

Proposition 5.7. *Let* $f : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$ *satisfy* (a)–(c) *from Theorem* 1.3. *Given* $0 < \alpha < 1$, $M_0 > 0$ *and* $1 < q < \frac{n+1}{n}$, *there exist two parameters* $\sigma = \sigma(n,L,\ell,\alpha,M_0,q) \in (0,\frac{1}{5})$ *as well as* $\widetilde{\varepsilon} = \widetilde{\varepsilon}(n,L,\ell,\alpha,M_0,q) \in (0,1)$ *such that every local* BD-*minimiser* $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ *of the functional* F *satisfies the following: If* $\mathrm{B}(x_0,R_0) \Subset \Omega$ *is an open ball with* $0 < R_0 \le 1$ *together with*

(5.42)
$$
\widetilde{\mathbf{E}}(u;x_0,R_0) \le \widetilde{\varepsilon}^2 \quad and \quad |(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}| \le M_0,
$$

then there holds

(5.43)
$$
\widetilde{\mathbf{E}}(u;x_0,\sigma R_0) \le \sigma^{1+\alpha} \widetilde{\mathbf{E}}(u;x_0,R_0).
$$

Proof. Let $\alpha \in (0,1)$ and $M_0 > 0$ be given. We estimate with $H := \mathrm{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}$, Lemma 2.10 and the shorthands $A_{R_0}^{\lessgtr} := \mathrm{B}(x_0,R_0) \cap \{|H| \lessgtr 1\}$

$$
\fint_{\mathrm{B}(x_0,R_0)} |H| \le \frac{1}{\omega_n R_0^n} \int_{A_{R_0}^{\le}} |H| + \frac{1}{\omega_n R_0^n} \int_{A_{R_0}^{>}} |H| = \frac{\mathscr{L}^n(A_{R_0}^{\le})}{\omega_n R_0^n} \fint_{A_{R_0}^{\le}} |H| + \frac{C}{R_0^n} \int_{A_{R_0}^{>}} V(|H|)
$$

$$
\le C \frac{\mathscr{L}^n(A_{R_0}^{\le})^{\frac{1}{2}}}{R_0^n} \left(\int_{\mathrm{B}(x_0,R_0)} V(|H|) \right)^{\frac{1}{2}} + \frac{C}{R_0^n} \int_{\mathrm{B}(x_0,R_0)} V(|H|)
$$

$$
\le C \left(\fint_{\mathrm{B}(x_0,R_0)} V(|H|) \right)^{\frac{1}{2}} + C \fint_{\mathrm{B}(x_0,R_0)} V(|H|) \le C(\sqrt{\widetilde{\mathbf{E}}(u;x_0,R_0)} + \widetilde{\mathbf{E}}(u;x_0,R_0)),
$$

where $C = C(n) > 0$. We may thus choose a preliminary $\widetilde{\varepsilon}_0 \in (0,1)$ such that $\widetilde{\mathbf{E}}(u;x_0,R_0) \le \widetilde{\varepsilon}_0^2$ and $|(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}| \le M_0$ imply (5.30) and (5.31). At this stage, for $1 < q < \frac{n+1}{n}$, (5.41) is available and therefore yields for $0 < \sigma < \frac{1}{5}$

(5.44)
$$
\widetilde{\mathbf{E}}(u;x_0,\sigma R_0) \le \left(\frac{C}{\sigma^{n+2}} \left(\widetilde{\mathbf{E}}(u;x_0,R_0) \right)^{q-1} + C\sigma^2 \right) \widetilde{\mathbf{E}}(u;x_0,R_0),
$$

where now[3] $C = C(n,M_0,L,\ell,q) > 0$. We subsequently choose $\sigma = \sigma(n,M_0,L,\ell,q,\alpha) > 0$ so small such that with the constant $C > 0$ from (5.44) there holds $2C\sigma^2 \le \sigma^{1+\alpha}$. We then put $\widetilde{\varepsilon} := \min\{\sigma^{\frac{n+4}{2(q-1)}}, \widetilde{\varepsilon}_0\}$. In turn, if $\widetilde{\mathbf{E}}(u;x_0,R_0) \le \widetilde{\varepsilon}^2$ and $|(\mathrm{E}u)_{\mathrm{B}(x_0,R_0)}| \le M_0$, then (5.44) gives

$$
\widetilde{\mathbf{E}}(u;x_0,\sigma R_0) \le (2C\sigma^2)\widetilde{\mathbf{E}}(u;x_0,R_0) \le \sigma^{1+\alpha} \widetilde{\mathbf{E}}(u;x_0,R_0),
$$

and this is precisely (5.46). The proof is complete. $\qquad\square$

---

[3] Note that the constant $C > 0$ in (5.41) depends on n,m,L,ℓ and q, but by (5.35), m depends on n and M_0 only.

5.4. **Iteration and Proof of Theorem 1.3.** To conclude the proof of Theorem 1.3, we need to iterate Proposition 5.7.

Corollary 5.8 (Iteration)**.** *Let* $f \colon \mathbb{R}^{n \times n}_{\mathrm{sym}} \to \mathbb{R}$ *satisfy* (a)–(c) *from Theorem* 1.3. *Given* $0 < \alpha < 1$ *and* $M_0 > 0$, *there exist* $\varepsilon = \varepsilon(n, L, \ell, \alpha, M_0) \in (0, 1)$ *and* $R_0 = R_0(n, L, \ell, M_0, \alpha) \in (0, 1)$ *such that every generalised local minimiser* $u \in \mathrm{BD}_{\mathrm{loc}}(\Omega)$ *of the functional F satisfies the following: If* $x_0 \in \Omega$ *and* $0 < R < R_0$ *are such that* $\mathrm{B}(x_0, R_0) \Subset \Omega$ *and*

$$(5.45) \qquad \widetilde{\mathbf{E}}(u; x_0, R) \leq \varepsilon^2 \quad and \quad |(\mathrm{E}u)_{\mathrm{B}(x_0, R)}| \leq \frac{M_0}{2},$$

then there holds

$$(5.46) \qquad \widetilde{\mathbf{E}}(u; x_0, r) \leq C \Big(\frac{r}{R} \Big)^{2\alpha} \widetilde{\mathbf{E}}(u; x_0, R) \qquad for\ all\ 0 < r \leq R.$$

Here, $C = C(n, L, \ell, \alpha, M_0) > 0$ *is a constant.*

The corollary is proved in a standard manner, the proof following, e.g., [46, Prop. 4.8] or [13, Lem. 5.8]; note that the dependence on q in Proposition 5.7 is removed by specialising, e.g., to $q = \frac{2n+1}{2n} \in (1, \frac{n+1}{n})$. Working from here, we can proceed to the

Proof of Theorem 1.3. Let $0 < \alpha < 1$ and $M > 0$ be given. We put $M_0 := 8 \max\{M, 1\}$. Then, by the previous corollary, there exist $\varepsilon = \varepsilon(n, L, \ell, \alpha, M_0) \in (0, 1)$ and $R_0 = R_0(n, L, \ell, M_0, \alpha) \in (0, 1)$ such that (5.45) implies (5.46). Within the framework of Theorem 1.3, we put $\varepsilon_M := \varepsilon^2 / 2^{2n+4}$ and let $0 < R < R_0$ be such that

$$(5.47) \qquad \widetilde{\mathbf{E}}(u; x_0, R) \leq \varepsilon_M = \frac{\varepsilon^2}{2^{2n+4}} \quad and \quad |(\mathrm{E}u)_{\mathrm{B}(x_0, R)}| \leq M(\leq \tfrac{1}{2}M_0).$$

Our aim is to show that with $R' := \frac{1}{2}R$ there holds $\widetilde{\mathbf{E}}(u; x, R') \leq \varepsilon^2$ and $|(\mathrm{E}u)_{\mathrm{B}(x, R')}| \leq \frac{1}{2}M_0$ for all $x \in \mathrm{B}(x_0, R')$. We have

$$\fint_{\mathrm{B}(x,R')} V(|\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x,R')}|) \, \mathrm{d}\mathscr{L}^n + \frac{|\mathrm{E}^s u|(\mathrm{B}(x,R'))}{\omega_n (R')^n}$$

$$\overset{(2.19)_3, V(\cdot) \leq |\cdot|}{\leq} 2 \fint_{\mathrm{B}(x,R')} V(|\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x,R')}|) \, \mathrm{d}\mathscr{L}^n + 3 \frac{|\mathrm{E}^s u|(\mathrm{B}(x,R'))}{\omega_n (R')^n}$$

$$\leq \frac{2 \cdot 2^{2n}}{\omega_n^2 R^{2n}} \int_{\mathrm{B}(x_0,R)} \int_{\mathrm{B}(x_0,R)} V(|\mathscr{E}u(y) - \mathscr{E}u(z)|) \, \mathrm{d}y \, \mathrm{d}z + 2^{n+2} \frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{\omega_n R^n}$$

$$\overset{(2.19)_3}{\leq} \frac{8 \cdot 2^{2n}}{\omega_n^2 R^{2n}} \int_{\mathrm{B}(x_0,R)} \int_{\mathrm{B}(x_0,R)} V(|\mathscr{E}u(y) - (\mathrm{E}u)_{\mathrm{B}(x_0,R)}|) \, \mathrm{d}y \, \mathrm{d}z + 2^{n+2} \frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{\omega_n R^n}$$

$$\overset{(5.47)}{\leq} 2^{2n+3} \frac{\varepsilon^2}{2^{2n+4}} < \varepsilon^2.$$

On the other hand, we have by (5.47) and $(2.19)_3$ in the third step

$$|(\mathrm{E}u)_{\mathrm{B}(x,R')}| \leq \left| \fint_{\mathrm{B}(x,R')} \mathscr{E}u \, \mathrm{d}\mathscr{L}^n \right| + \frac{|\mathrm{E}^s u|(\mathrm{B}(x,R'))}{\omega_n (R')^n}$$

$$\leq \left| \fint_{\mathrm{B}(x,R')} \mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)} \, \mathrm{d}\mathscr{L}^n \right| + \frac{|\mathrm{E}^s u|(\mathrm{B}(x,R'))}{\omega_n (R')^n} + |(\mathrm{E}u)_{\mathrm{B}(x_0,R)}|$$

$$\leq 2^n \Big(\frac{1}{\sqrt{2}-1} \fint_{\mathrm{B}(x_0,R)} V(|\mathscr{E}u - (\mathrm{E}u)_{\mathrm{B}(x_0,R)}|) \, \mathrm{d}\mathscr{L}^n \Big)^{\frac{1}{2}} + 2^n \frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,R))}{\omega_n R^n} + M$$

$$\leq \frac{\varepsilon}{4\sqrt{\sqrt{2}-1}} + \frac{\varepsilon^2}{2^{n+4}} + M \leq M + 1$$

having used that $\varepsilon \in (0,1)$ in the ultimate step. Since $M + 1 \leq 2\max\{M,1\} \leq \frac{M_0}{2}$, we thus obtain by (5.47) and (5.46) that for all $x \in \mathrm{B}(x_0, R')$ and all $0 < r < R'$ there holds

$$\widetilde{\mathbf{E}}(u;x,r) \leq C(n,L,\ell,\alpha,M_0)\left(\frac{r}{R'}\right)^{2\alpha}\widetilde{\mathbf{E}}(u;x,R').$$

Working from here, we first deduce by sending $r \searrow 0$ that $\mathrm{E}^s u \equiv 0$ in $\mathrm{B}(x_0, R')$. Therefore, setting $G(t) := \min\{t, t^2\}$ for $t \geq 0$, we find by $(2.19)_1$ and Jensen's inequality

$$(\sqrt{2}-1)G(\Upsilon') := (\sqrt{2}-1)G\left(\fint_{\mathrm{B}(x,r)}|\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x,r)}|\,\mathrm{d}\mathscr{L}^n\right) \leq C(n,L,\ell,\alpha,M_0)\left(\frac{r}{R'}\right)^{2\alpha}\varepsilon^2,$$

for all $0 < r < R'$, with Υ' defined in the obvious manner. Now, if $0 \leq \Upsilon' \leq 1$, the previous estimate yields $|\Upsilon'|^2 \leq C(\frac{r}{R'})^{2\alpha}$ whereas if $|\Upsilon'| > 1$, we use $(\frac{r}{R'})^{2\alpha} \leq (\frac{r}{R'})^\alpha$ to infer that

$$\frac{1}{r^\alpha}\fint_{\mathrm{B}(x,r)}|\mathscr{E}u - (\mathscr{E}u)_{\mathrm{B}(x,r)}|\,\mathrm{d}\mathscr{L}^n \leq \frac{C(n,L,\ell,\alpha,M_0)}{(R')^\alpha}$$

for all $0 < r < R'$. Now, by the Campanato-Meyers characterisation of Hölder continuity, this implies that $\mathscr{E}u$ is of class $\mathrm{C}^{0,\alpha}$ in $\mathrm{B}(x_0, R')$. Now we use $\mathrm{C}^{0,\alpha} \simeq \mathscr{L}^{2,n+2\alpha}$ with the CAMPANATO spaces $\mathscr{L}^{p,\lambda}$ (cf. [42, Thm. 2.9]) and Proposition 2.8(c) with $\psi(t) = t^2$ to find that $\mathrm{D}u$ is of class $\mathrm{C}^{0,\alpha}$ in $\mathrm{B}(x_0, R')$, too.

Since $V(\cdot) \leq |\cdot|$ and with the above choices of ε_M, R_0, (1.10) implies (5.45). Hence, to conclude the proof of the theorem, we note that by the Lebesgue differentiation theorem for Radon measures, for $\mathscr{L}^n$-a.e. $x_0 \in \Omega$ there exists $z \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ such that

$$(5.48) \qquad \lim_{r \searrow 0}\fint_{\mathrm{B}(x_0,r)}|\mathscr{E}u - z|\,\mathrm{d}\mathscr{L}^n + \frac{|\mathrm{E}^s u|(\mathrm{B}(x_0,r))}{\omega_n r^n} = 0.$$

Thus, for such x_0, there exists $\delta > 0$ with $\sup_{0<r<\delta}|(\mathrm{E}u)_{\mathrm{B}(x_0,r)}| =: M < \infty$. Applying the foregoing to the number M and invoking (5.48), the conclusion of the theorem follows immediately and the proof is complete. $\qquad\square$

We concude with the following

Remark 5.9. In the BV-case as considered by KRISTENSEN and the author [46], different Fubini-type properties needed to be invoked to deal with $n = 2$ and $n \geq 3$. Starting from the fact that for BV-maps the tangential derivatives of u on $\partial\mathrm{B}(x_0,t)$ for $\mathscr{L}^1$-a.e. $t > 0$ are finite Radon measures themselves, the approach in [46] is to embed $\mathrm{BV}(\partial\mathrm{B}(x_0,t);\mathbb{R}^N)$ into higher fractional Sobolev spaces. If $n = 2$, spheres are one-dimensional manifolds, and here Remark 2.3 excludes the relevant embeddings. This forces to argue via Besov-Nikolskiĭ spaces in the full gradient, strongly quasiconvex case for $n = 2$. However, the approach as outlined above for BD equally works in the easier BV-situation, too, and thus yields a unifying method for all $n \geq 2$.

5.5. **Remarks and Extensions.**
In this concluding section, we discuss some aspects, extensions and limitations of the results presented so far.

We begin by noting that, under the assumptions of Theorem 1.3, we can actually establish $\mathrm{C}^{2,\alpha}$-partial regularity of generalised minima. Namely, letting $x_0 \in \Omega_u$, we have $u \in \mathrm{C}^{2,\alpha}(\mathrm{B}(x_0,r);\mathbb{R}^n)$ for some $r > 0$ and all $0 < \alpha < 1$. This is a consequence of Schauder estimates based on the $\mathrm{C}^{1,\alpha}$-regularity of u in a neighbourhood of x_0; choosing $|h|$ small enough, we consider the finite differences $\tau_{s,h}\varepsilon(u)(x) := \varepsilon(u)(x + he_s) - \varepsilon(u)(x)$, where x belongs to a suitable neighbourhood of x_0 and e_s is the s-th unit vector. We then set

$$\mathscr{Q}(x)[\xi,\eta] := \int_0^1 \langle f''(\varepsilon(u)(x) + t\tau_{s,h}\varepsilon(u)(x))\xi,\eta\rangle\,\mathrm{d}t, \qquad \xi,\eta \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

By condition (a) from Theorem 1.3 and the $\mathrm{C}^{1,\alpha}$-partial regularity (for any $0 < \alpha < 1$) established above, $\mathscr{Q} \in \mathrm{C}^{0,\alpha}(U; \mathbb{S}(\mathbb{R}^{n\times n}_{\mathrm{sym}}))$ for some open neighbourhood U of x_0 and any $0 < \alpha < 1$. Possibly diminishing U, we infer similarly as to (5.9) that $\mathscr{Q}$ is uniformly symmetric Legendre-Hadamard in U; i.e, there exists a constant $\lambda > 0$ such that $\mathscr{Q}(x)[a \odot b, a \odot b] \geq \lambda|a \odot b|^2$ for

all $a, b \in \mathbb{R}^n$ and all $x \in U$ together with $\sup_{x \in U} |\mathscr{Q}(x)| < \infty$. Working from here, it is not too difficult to establish an inequality of Garding type for some $r > 0$ suitably small: There exist $\gamma_1, \gamma_2 > 0$ such that there holds

$$\int_{\mathrm{B}(x_0, r)} \mathscr{Q}(x)[\varepsilon(\varphi), \varepsilon(\varphi)] \, \mathrm{d}x \geq \int_{\mathrm{B}(x_0, r)} \gamma_1 |\varepsilon(\varphi)|^2 - \gamma_2 |\varphi|^2 \, \mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{W}_0^{1,\infty}(\mathrm{B}(x_0, r); \mathbb{R}^n).$$

On the other hand, since u is a minimiser, we deduce the Euler-Lagrange equation

$$\int_{\mathrm{B}(x_0, r)} \langle f'(\varepsilon(u)), \varepsilon(\varphi) \rangle \, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{LD}_0(\mathrm{B}(x_0, r)).$$

At this stage, picking an arbitrary localisation function $\rho \in \mathrm{C}_c^\infty(\mathrm{B}(x_0, r); [0, 1])$, we may test the preceding equation with $\varphi = \tau_{s, -h}(\rho^2 \tau_{s, h} u)$ for $|h|$ suitably small. Here $\tau_{s, -h} v(x) := v(x - he_s) - v(x)$ denotes the backward finite difference. As a consequence, we obtain that

$$\int_{\mathrm{B}(x_0, r)} \mathscr{Q}(x)[\tau_{s, h} \varepsilon(u), \rho^2 \tau_{s, h} \varepsilon(u) + 2\rho \nabla\rho \odot \tau_{s, h} u] \, \mathrm{d}x = 0$$

holds for any $s \in \{1, ..., n\}$. Then it is routine to conclude by the above Garding-type inequality and Korn's inequality that u is of class $\mathrm{W}^{2,2}$ in a suitable neighbourhood of x_0. Diminishing $r > 0$ if necessary, we may then assume that

$$(5.49) \qquad \int_\Omega \langle f''(\varepsilon(u)) \partial_s \varepsilon(u), \varepsilon(\varphi) \rangle \, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in \mathrm{C}_c^1(\mathrm{B}(x_0, r); \mathbb{R}^n).$$

At this stage, we invoke a similar argument as in Proposition 5.3 to reduce to the SCHAUDER estimates [41, Ch. III, Thm. 3.2] to conclude that $\partial_s \varepsilon(u)$ is of class $\mathrm{C}^{0,\alpha}$ in a fixed neighbourhood of x_0. Switching to the CAMPANATO-MEYERS characterisation of Hölder continuity and employing Proposition 2.8(c), $\partial_s \mathrm{D}u \in \mathrm{C}^{0,\alpha}$ in an open neighbourhood of x_0 for any $s \in \{1, ..., n\}$.

An analogous regularity theory can be set up when x-dependent integrands $f \colon \Omega \times \mathbb{R}_{\mathrm{sym}}^{n \times n} \to \mathbb{R}$ are considered, and we refer the reader to the corresponding statements in [46, Sec. 6]; these follow in an analogous way once the regularity results from Theorem 1.3 are available. However, the case of *fully* non-autonomous integrands $f \colon \Omega \times \mathbb{R}^n \times \mathbb{R}_{\mathrm{sym}}^{n \times n} \ni (x, y, z) \mapsto f(x, y, z)$ comes along with two major difficulties. First, to the best of the author's knowledge, there is no integral representation of the relaxed functional available at present; the arguments of RINDLER [65] do not seem to easily generalise to this situation. In contrast to (1.3), the definition of generalised minima then must be given directly by the Lebesgue-Serrin-type extension. To then access the Euler-Lagrange equations satisfied by the respective BD-minimisers, it is necessary to employ a careful approximation procedure. This in principle being possible, we would still need a higher integrability result on the gradients of BD-minima as it is usually required (cf. [42, Thm. 9.5 ff.]). In the quasiconvex, superlinear growth context, the latter is obtained as a consequence of the Caccioppoli inequality of the second kind in conjunction with the Sobolev inequality. In this situation, the Gehring lemma then boosts the so derived reverse Hölder inequality with increasing supports to the higher integrability of the gradients. In the linear growth situation, working from the Caccioppoli-type inequality strictly requires a *sublinear Sobolev inequality*, the unconditional availability of which being ruled out by a counterexample due to BUCKLEY & KOSKELA [18]. This is an important issue, as otherwise we would immediately obtain that BD-minima belonged to $\mathrm{W}_{\mathrm{loc}}^{1,q}$ for some $q > 1$, a fact which would simplify several stages of the above proof. A similar issue had been identified by ANZELLOTTI & GIAQUINTA [9, Sec. 6] within the framework of convex full gradient functionals. However, note that if $f \colon \Omega \times \mathbb{R}^n \times \mathbb{R}_{\mathrm{sym}}^{n \times n} \to \mathbb{R}$ satisfies a splitting condition $f(x, y, z) = f_1(x, z) + f_2(x, y)$ for some strongly symmetric quasiconvex integrand $f_1 \colon \Omega \times \mathbb{R}_{\mathrm{sym}}^{n \times n} \to \mathbb{R}$ of linear growth and $f_2 \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ being convex and of at most $\frac{n}{n-1}$-growth in the second variable, then suitable regularity results *can be formulated*. SCHMIDT [68] provides an interesting alternative of a partial regularity proof for convex, fully non-autonomous integrands of (super)quadratic growth that does not utilise Gehring's lemma.

The drawback here is that does not seem to generalise easily to the quasiconvex situation with (super)linear growth; even if it would, it needed to be compatible with the above proof scheme.

Lastly, let us address the possibility of giving Hausdorff dimension bounds on Σ_u in the framework of Theorems 1.2 or 1.3. Whereas in the convex context higher differentiability of minima can be invoked to establish $\dim_{\mathscr{H}}(\Sigma_u) < n$ (also see [42, 57, 58] and [25, 43, 44, 45] in the symmetric gradient situation), such methods rely exclusively on the Euler-Lagrange system and thus do not apply to the (symmetric) quasiconvex situation. In this case, one option is to apply the set porosity approach employed by KRISTENSEN & MINGIONE [53] for Lipschitzian minima. In the framework of Theorem 1.2 for $p \geq 2$, this directly yields the dimension reduction for $W^{1,\infty}$-regular minima. The method is likely to generalise to $1 \leq p < \infty$; however, to the best of the author's knowledge, the method is not known to yield a dimension reduction for $W^{1,s}$-regular minima even with large $s > p \geq 1$; neither is it clear how to obtain such a higher local gradient integrability in the (strongly) quasiconvex context beyond the usual GEHRING-type improvement. Indeed, for $p > 1$, *some* higher gradient integrability follows by GEHRING's lemma in conjunction with the Caccioppoli inequality, but by the above discussion even this is unclear in the linear growth context. We intend to tackle questions of this sort in the future.

6. APPENDIX

6.1. **Existence of** BD**-minima.** Implicitly used in the main part, we now briefly justify the existence of generalised minima for the Dirichlet problem (1.1) in the sense of (1.3), now being subject to the strong symmetric quasiconvexity of $f \in C(\mathbb{R}^{n\times n}_{\mathrm{sym}})$, and gather some consequences. This program is somewhat analogous to [17, Thm. 5.3] where, however, a different coerciveness condition was employed. We hereafter let $u_0 \in \mathrm{LD}(\Omega)$ be a given Dirichlet datum and $f \in C(\mathbb{R}^{n\times n}_{\mathrm{sym}})$ a strongly symmetric quasiconvex integrand satisfying both (1.9) and the linear growth assumption (LG). Our objective is to establish (with the notation of (1.1) ff.)

$$\tag{6.1} \inf_{\mathscr{D}_{u_0}} F = \min_{\mathrm{BD}(\Omega)} \overline{F}_{u_0},$$

particularly asserting the existence of BD-minimisers. Toward the latter, we note that because F is strongly symmetric quasiconvex, we have for all $\varphi \in C^\infty_c(\Omega;\mathbb{R}^n)$

$$f(0) = f(0) - \ell V(0) \leq \fint_\Omega f(\varepsilon(\varphi)) - \ell V(\varepsilon(\varphi))\,\mathrm{d}x,$$

as follows easily by passing from $Q = (0,1)^n$ to general open domains Ω. Thus, by Lipschitz continuity of f,

$$\tag{6.2} \begin{aligned} f(0)\mathscr{L}^n(\Omega) + \ell\int_\Omega V(\varepsilon(\varphi))\,\mathrm{d}x &\leq \int_\Omega f(\varepsilon(\varphi)) - f(\varepsilon(u_0 + \varphi))\,\mathrm{d}x + \int_\Omega f(\varepsilon(u_0 + \varphi))\,\mathrm{d}x \\ &\leq c(L)\int_\Omega |\varepsilon(u_0)|\,\mathrm{d}x + \int_\Omega f(\varepsilon(u_0 + \varphi))\,\mathrm{d}x. \end{aligned}$$

At this stage, we pick an open and bounded Lipschitz set $\widetilde{\Omega} \subset \mathbb{R}^n$ such that $\Omega \Subset \widetilde{\Omega}$ and find, following the discussion in Section 2.2, $\overline{u}_0 \in \mathrm{LD}_0(\widetilde{\Omega})$ such that $\overline{u}_0|_\Omega = u_0$. We then put, for $v \in \mathrm{BD}(\Omega)$

$$\tag{6.3} \widetilde{v} := \begin{cases} v & \text{in } \Omega, \\ \overline{u}_0 & \text{in } \widetilde{\Omega} \setminus \overline{\Omega}. \end{cases}$$

Since $\partial\Omega$ is Lipschitz and $\overline{u}_0 \in \mathrm{LD}(\widetilde{\Omega})$, $\widetilde{v} \in \mathrm{BD}(\widetilde{\Omega})$. Hence, we have for all $\varphi \in C^\infty_c(\Omega;\mathbb{R}^n)$

$$\begin{aligned} \ell\int_{\widetilde{\Omega}} V(\varepsilon(\overline{u}_0 + \varphi))\,\mathrm{d}x - \ell\int_{\widetilde{\Omega}\setminus\Omega} V(\varepsilon(\overline{u}_0))\,\mathrm{d}x &= \ell\int_\Omega V(\varepsilon(u_0 + \varphi)) - V(\varepsilon(\varphi))\,\mathrm{d}x + \ell\int_\Omega V(\varepsilon(\varphi))\,\mathrm{d}x \\ &\overset{(6.2)}{\leq} C(\ell, L, V)\int_\Omega |\varepsilon(u_0)| - f(0)\mathscr{L}^n(\Omega) + \int_\Omega f(\varepsilon(u_0 + \varphi))\,\mathrm{d}x \end{aligned}$$

$$\overset{(\mathrm{LG})}{\leq} C(\ell,L,V,\mathscr{L}^n(\widetilde{\Omega}))\Big(\int_{\widetilde{\Omega}}|\varepsilon(\overline{u}_0)|\,\mathrm{d}x+1\Big)+\int_{\widetilde{\Omega}}f(\varepsilon(\overline{u}_0+\varphi))\,\mathrm{d}x.$$

At this stage, let $v\in\mathrm{BD}(\Omega)$ be arbitrary and pick, due to Lemma 2.1, a sequence $(v_j)\subset u_0+\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that $\widetilde{v}_j\to\widetilde{v}$ area-strictly on $\mathrm{BD}(\widetilde{\Omega})$. Since for every $j\in\mathbb{N}$, $\widetilde{v}_j$ is of the form $\overline{u}_0+\varphi_j$ with some $\varphi_j\in\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$, we obtain

$$\ell\int_{\widetilde{\Omega}}V(\varepsilon(\widetilde{v}_j))\,\mathrm{d}x-\ell\int_{\widetilde{\Omega}\setminus\Omega}V(\varepsilon(\overline{u}_0))\,\mathrm{d}x\leq C(\ell,L,V,\mathscr{L}^n(\widetilde{\Omega}))\Big(\int_{\widetilde{\Omega}}|\varepsilon(\overline{u}_0)|\,\mathrm{d}x+1\Big)+\int_{\widetilde{\Omega}}f(\varepsilon(\widetilde{v}_j))\,\mathrm{d}x.$$

Since f is symmetric quasiconvex, it is symmetric rank-one convex in the sense as specified in Section 2.3. Therefore, Lemma 2.6 (which precisely yields continuity of the associated integral functionals for symmetric rank-one convex integrands) and the very definition of V yield by passing $j\to\infty$

$$\ell\int_{\widetilde{\Omega}}V(\mathrm{E}\widetilde{v})-\ell\int_{\widetilde{\Omega}\setminus\Omega}V(\varepsilon(\overline{u}_0))\,\mathrm{d}x\leq C(\ell,L,V,\mathscr{L}^n(\widetilde{\Omega}))\Big(\int_{\widetilde{\Omega}}|\varepsilon(\overline{u}_0)|\,\mathrm{d}x+1\Big)+\int_{\widetilde{\Omega}}f(\mathrm{E}\widetilde{v}).$$

Enlarging the constant $C>0$ from the previous inequality, by definition of $\widetilde{v}$ we thereby obtain

(6.4)
$$\ell\int_{\Omega}V(\mathrm{E}v)+\ell\int_{\Omega}|\mathrm{Tr}_{\partial\Omega}(u_0-v)\odot\nu_{\partial\Omega}|\,\mathrm{d}\mathscr{H}^{n-1}-C(\ell,L,V,\mathscr{L}^n(\widetilde{\Omega}))\Big(\int_{\widetilde{\Omega}}|\varepsilon(\overline{u}_0)|\,\mathrm{d}x+1\Big)$$
$$\overset{(\mathrm{LG})}{\leq}\int_{\Omega}f(\mathrm{E}v)+\int_{\partial\Omega}f^\infty\big(\mathrm{Tr}_{\partial\Omega}(u_0-v)\odot\nu_{\partial\Omega}\big)\,\mathrm{d}\mathscr{H}^{n-1}=\overline{F}_{u_0}[v].$$

Since $|\cdot|\leq V(\cdot)+1$, this proves that $\overline{F}_{u_0}$ is bounded below on $\mathrm{BD}(\Omega)$. Let $(u_j)\subset\mathrm{BD}(\Omega)$ be a minimising sequence for $\overline{F}_{u_0}$, i.e., $\overline{F}_{u_0}[u_j]\to\inf_{\mathrm{BD}(\Omega)}\overline{F}_{u_0}$. Then (u_j) is bounded in $\mathrm{BD}(\Omega)$ and we may extract a non-relabeled subsequence and find some $u\in\mathrm{BD}(\Omega)$ such that $u_j\overset{*}{\rightharpoonup}u$ in $\mathrm{BD}(\Omega)$ (and hence $\widetilde{u}_j\overset{*}{\rightharpoonup}\widetilde{u}$ in $\mathrm{BD}(\widetilde{\Omega})$). Cancelling the integrals over $\widetilde{\Omega}\setminus\overline{\Omega}$ and as a consequence of Theorem 2.5, $\overline{F}_{u_0}[u]\leq\liminf_{j\to\infty}\overline{F}_{u_0}[u_j]=\inf_{\mathrm{BD}(\Omega)}\overline{F}_{u_0}$. Hence, u is a BD-minimiser in the sense of (1.3).

We come to (6.1). Since there holds $\mathscr{D}_{u_0}\subset\mathrm{BD}(\Omega)$ and $\overline{F}_{u_0}|_{\mathscr{D}_{u_0}(\Omega)}=F$ on $\mathscr{D}_{u_0}$, we obtain '$\geq$' in (6.1). For the other direction, pick a BD-minimiser $u\in\mathrm{BD}(\Omega)$ for F, its existence having been established above. Choosing an extension $\overline{u}_0$ of the Dirichlet datum u_0 as above and defining $\widetilde{u}$ via (6.3), we invoke Lemma 2.1 to obtain a sequence $(u_j)\subset u_0+\mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)$ such that $\widetilde{u}_j\to\widetilde{u}$ area-strictly in $\mathrm{BD}(\widetilde{\Omega})$. Then, again by Lemma 2.6, $\overline{F}_{u_0}[u_j]\to\overline{F}_{u_0}[u]$ as $j\to\infty$. Thus, since $\widetilde{u}_j\in\mathscr{D}_{u_0}$ and $\overline{F}_{u_0}[u_j]=F[u_j]$ for all $j\in\mathbb{N}$,

$$\inf_{\mathscr{D}_{u_0}}F\leq\lim_{j\to\infty}\overline{F}_{u_0}[u_j]=\overline{F}_{u_0}[u]=\min_{\mathrm{BD}(\Omega)}\overline{F}_{u_0}[u].$$

Since we already established that $\min_{\mathrm{BD}(\Omega)}\overline{F}_{u_0}\leq\inf_{\mathscr{D}_{u_0}}F$, the proof of (6.1) is complete.

6.2. **Auxiliary Estimates on the V_p-functions.** In this section we provide the proof of the auxiliary estimation (3.9) that helped to establish a particular form of a Korn-type inequality; recall that now $1<p<2$. The first uniform comparability assertion of (3.9) is a basic property of shifted N-functions, cf. [31, Def. 2 and Sec. 2]. We thus begin by showing that ψ given by (3.8) satisfies the conditions of Lemma 2.9 together with the second uniform comparability of (3.9). This means

(6.5)
$$c\psi'(t)\leq\psi''(t)t\leq C\psi'(t),$$
$$c(1+a+t)^{p-2}t^2\leq\psi''(a+t)t^2\leq C(1+a+t)^{p-2}t^2$$

for some $0<c\leq C<\infty$ independent of $a\geq0$ and $t>0$. We start with $(6.5)_2$, and recall that $1<p<2$ throughout this section. To this end, note that for $t>0$

(6.6)
$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)=(p-2)(1+t)^{p-3}t^2+2(1+t)^{p-2}t,$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\psi(t)=(p-2)(p-3)(1+t)^{p-4}t^2+4(p-2)(1+t)^{p-3}t+2(1+t)^{p-2}.$$

Since $1 < p < 2$, the second term on the right-hand side of $(6.6)_2$ is negative. Therefore,

$$\psi''(a+t)t^2 \leq (p-2)(p-3)(1+a+t)^{p-4}(a+t)^2t^2 + 2(1+a+t)^{p-2}t^2$$
$$\leq ((p-2)(p-3)+2)(1+a+t)^{p-2}t^2,$$

establishing the upper bound asserted by (6.5). The lower bound requires a refined argument. Since $p > 1$, $c_p := p^2 - p$ is strictly positive. We write for $t > 0$

$$\frac{\psi''(a+t)t^2}{(1+a+t)^{p-2}t^2} = \frac{(p-2)(p-3)(1+a+t)^{p-4}(a+t)^2t^2}{(1+a+t)^{p-2}t^2} + \frac{4(p-2)(1+a+t)^{p-3}(t+a)t^2}{(1+a+t)^{p-2}t^2}$$
$$+ \frac{2(1+a+t)^{p-2}t^2}{(1+a+t)^{p-2}t^2}$$
$$= 2 + (p-2)\left[(p-3)\left(\frac{a+t}{1+a+t}\right)^2 + 4\left(\frac{a+t}{1+a+t}\right)\right]$$

We claim that the ultimate term is larger or equal than c_p. Put $\vartheta \colon \mathbb{R} \ni z \mapsto 2 + (p-2)((p-3)z^2 + 4z)$. Since $1 < p < 2$, this function has a global minimum at $z_0 = \frac{2}{3-p}$ which, by $p > 1$, satisfies $z_0 > 1$. Hence, for all $z \in (0,1)$, $\vartheta(z) \geq \vartheta(1) = p^2 - p = c_p > 0$, and the lower bound of (6.5) follows because of $(a+t)/(1+a+t) \in (0,1)$.

We turn to the third uniform comparability assertion of (3.9), which is equivalent to the existence of constants $0 < c \leq C < \infty$ such that

$$(6.7) \qquad c(1+t^2+a^2)^{\frac{p-2}{2}}t^2 \leq (1+t+a)^{p-2}t^2 \leq C(1+t^2+a^2)^{\frac{p-2}{2}}t^2$$

holds for all $a, t \geq 0$. First note that

$$\sqrt{1+t^2+a^2} \leq \sqrt{(1+t+a)^2} = 1+t+a$$

so that, because of $1 < p < 2$, $(1+t+a)^{p-2} \leq (1+t^2+a^2)^{\frac{p-2}{2}}$, and so the upper bound in (6.7) follows. For the lower bound note that, because of Young's inequality

$$1+t+a = \sqrt{(1+t+a)^2} \leq \sqrt{8+8t^2+8a^2} \leq \sqrt{8}\sqrt{1+t^2+a^2},$$

thereby establishing the lower bound in (6.7); for the latter estimate, we could have alternatively argued by virtue of Lemma 2.10, cf. $(2.19)_1$. We now turn to $(6.5)_1$. Setting $a = 0$ in $(6.5)_2$, $(6.5)_1$ is obviously equivalent to

$$(6.8) \qquad\qquad \psi'(t) \simeq (1+t)^{p-2}t.$$

By $(6.6)_1$ and $1 < p < 2$, we have $\psi'(t) \leq 2(1+t)^{p-2}t$ for all $t > 0$. On the other hand, for $t > 0$,

$$\frac{\psi'(t)}{(1+t)^{p-2}t} = \frac{(p-2)(1+t)^{p-3}t^2 + 2(1+t)^{p-2}t}{(1+t)^{p-2}t} = (p-2)\frac{t}{1+t} + 2 \geq p.$$

The proof of (3.9) is complete.

REFERENCES

[1] Acerbi, E.; Fusco, N.: Semicontinuity problems in the calculus of variations. Arch. Rational Mech. Anal. 86 (1984), no. 2, 125–145.

[2] Acerbi, E.; Fusco, N.: A regularity theorem for minimizers of quasiconvex integrals. Arch. Rational Mech. Anal. 99 (1987), no. 3, 261–281.

[3] Acerbi, E.; Mingione, G.: Regularity results for stationary electro-rheological fluids. Arch. Rat. Mech. Anal. 164, 213– 259 (2002).

[4] Ambrosio, L.; Coscia, A.; Dal Maso, G.: Fine properties of functions with bounded deformation. (English summary) Arch. Rational Mech. Anal. 139 (1997), no. 3, 201–238.

[5] Alberti, G.: Rank one property for derivatives of functions with bounded variation, Proc. Roy. Soc. Edinburgh Sect. A 123 (1993), 239–274.

[6] Ambrosio, L.; Dal Maso, G.: On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasi–convex integrals. J. Funct. Anal. 109 (1992), 76–97.

[7]   Ambrosio, L.; Fusco, N.; Pallara, D.: Functions of bounded variation and free discontinuity problems. Oxford
      Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[8]   Anzellotti, G.; Giaquinta, M.: Existence of the displacements field for an elastoplastic body subject to Hencky's
      law and Von Mises yield condition. Manuscripta Mathematica, 32(1-2) (1980), 101-136.

[9]   Anzellotti, G.; Giaquinta, M.: Convex functionals and partial regularity. Arch. Rational Mech. Anal. 102 (1988),
      no. 3, 243–272.

[10] Arroyo-Rabasa, A.; De Philippis, G.; Rindler, F.: Lower semicontinuity and relaxation of linear-growth integral
      functionals under PDE constraints, arXiv:1701.02230 (2017), 1–32.

[11] Babadjian, J.–F.: Traces of functions of bounded deformation. Indiana Univ. Math. J. 64 (2015), no. 4, 1271–1290.

[12] Barroso, A.C.; Fonseca, I.; Toader, R.: A relaxation theorem in the space of functions of bounded deformation.
      Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 29 (2000), no. 1, 19–49.

[13] Beck, L.: Elliptic Regularity - A first course. Lecture Notes of the Italian Mathematical Union (19), 2016, Springer.

[14] Bildhauer, M.: Convex Variational Problems – Linear, Nearly Linear and Anisotropic Growth Conditions. Lecture
      Notes in Mathematics 1818, Springer Verlag 2003.

[15] Bourgain, J.; Brezis, H.; Mironescu, P.: $H^{\frac{1}{2}}$ maps with values into the circle: minimal connections, lifting, and the
      Ginzburg-Landau equation. Publ. Math. Inst. Hautes Études Sci. No. 99 (2004), 1–115.

[16] Breit, D., Diening, L.: Sharp Conditions for Korn Inequalities in Orlicz Spaces. J. Math. Fluid Mech. 14 (2012),
      565–573

[17] Breit, D.; Diening, L.; Gmeineder, F.: On the Trace Operator for Functions of bounded $\mathbb{A}$-Variation. Anal. PDE.
      13(2020), pp. 559–594.

[18] Buckley, S.M.; Koskela, P.: Sobolev-Poincaré inequalities for $p < 1$. Indiana Univ. Math. J. 43 (1994), no. 1,
      221–240.

[19] Carozza, M.; Fusco, N.; Mingione, G.: Partial regularity of minimizers of quasiconvex integrals with subquadratic
      growth. Ann. Mat. Pura Appl. (4) 175 (1998), 141–164.

[20] Christiansen, E.; Matthies, E.; Strang, G.: The saddle point of a differential program, in 'Energy methods in finite
      element analysis', volume dedicated to Professor Fraeijs de Veubeke; Glowinski, R., Rodin, E., Zienkiewicz, O.C.,
      eds. New York: John Wiley 1979.

[21] Ciarlet, P.G.: Inequalities of Korn's type on surfaces, in Il Problema di de Saint-Venant : Aspetti Teorici e Ap-
      plicativi, pp. 105-134, Atti dei Convegni Lincei 140, Accademia Nazionale dei Lincei, Roma, 1998.

[22] Ciarlet, P.G.; Gratie, L.; Mardare, L.: A nonlinear Korn inequality on a surface, J. Math. Pures Appl. 85 (2006),
      2-16.

[23] Ciarlet P.G.; Mardare, L.: Nonlinear Korn inequalities, J. Math. Pures Appl. 104 (2015), 1119-1134.

[24] Conti, S., Faraco, D., and Maggi, F.: A new approach to counterexamples to L 1 estimates: Korn's inequality,
      geometric rigidity, and regularity for gradients of separately convex functions. Archive for rational mechanics and
      analysis, 175(2), 287-300.

[25] Conti, S.; Focardi, M., Iurlano, F.: A note on the Hausdorff dimension of the singular set of solutions to elasticity
      type systems. Comm. Contemp. Math., Vol. 21, No. 06, 1950026 (2019).

[26] Dacorogna, B.: Weak continuity and weak lower semicontinuity of nonlinear func- tionals, Lecture Notes in
      Mathematics, vol. 922, Springer-Verlag, Berlin-New York, 1982.

[27] Dal Maso G., Fonseca I., Leoni G., Morini M.: Higher order quasiconvexity reduces to quasiconvexity. Arch.
      Ration. Mech. Anal. 171 (2004), 55-81.

[28] De Philippis, G.; Rindler, F.: On the structure of $\mathscr{A}$—free measures and applications. Ann. of Math. (2) 184
      (2016), no. 3, 1017–1039.

[29] De Philippis, G.; Rindler, F.: Characterization of generalized Young measures generated by symmetric gradients,
      Arch. Ration. Mech. Anal. 224 (2017) 1087–1125.

[30] Diening, L.; Ettwein, F.: Fractional estimates for non-differentiable elliptic systems with general growth. Forum
      Mathematicum, (2008) no. 3, 523-556.

[31] Diening, L.; Lengeler, D.; Stroffolini, B.; Verde, A.: Partial Regularity for Minimizers of Functionals with General
      Growth. SIAM J. Math. Anal. (2012), Vol. 44, No. 5, pp. 3594–3616.

[32] Duzaar, F.; Grotowski, J.F.: Optimal interior partial regularity for nonlinear elliptic systems: the method of A-
      harmonic approximation. Manuscripta Math. 103 (2000), no. 3, 267–298.

[33] Duzaar, F.; Steffen, K.: Optimal interior and boundary regularity for almost minimizers to elliptic variational
      integrals. J. Reine Angew. Math. 546 (2002), 73–138.

[34] Duzaar, F.; Grotowski, J.F.; Kronz, M.: Regularity of almost minimizers of quasi-convex variational integrals with
      subquadratic growth. Ann. Mat. Pura Appl. (4) 184 (2005), no. 4, 421–448.

[35] Ekeland, I.: On the variational principle. J. Math. Anal. Appl. 47 (1974), 324–353.

[36] Evans, L.C.: Quasiconvexity and partial regularity in the calculus of variations. Arch. Rational Mech. Anal. 95
      (1986), no. 3, 227–252.

[37] Fonseca, I.; Müller, S.: Relaxation of quasiconvex functionals in BV$(\Omega; \mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$, Arch. Rat.
      Mech. Anal. 123 (1993), 1–49.

[38] Fonseca, I.; Müller, S.: $\mathscr{A}$-quasiconvexity, lower semicontinuity, and Young measures. SIAM J. Math. Anal. 30 (1999), no. 6, 1355–1390.

[39] Friedrichs, K.: On the boundary value problems of the theory of elasticity and Korn's inequality. Ann. Math. 48(2), 441–471 (1947).

[40] Fuchs, M.; Seregin, G.: Variational methods for problems from plasticity theory and for generalized Newtonian fluids. Lecture Notes in Mathematics, 1749. Springer-Verlag, Berlin, 2000. vi+269 pp.

[41] Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983. vii+297 pp.

[42] Giusti, E.: Direct methods in the calculus of variations. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[43] Gmeineder, F.: Symmetric-convex functionals of linear growth. J. Ell. Par. Equations 2(1–2), 59–71, 2016.

[44] Gmeineder, F.: The regularity of minima for variational problems on BD. Arch. Ration. Mech. Anal., September 2020, Vol. 237, Issue 3, 1099–1171.

[45] Gmeineder, F.; Kristensen, J.: Sobolev regularity for convex functionals on BD. J. Calc. Var. (2019) 58:56.

[46] Gmeineder, F.; Kristensen, J.: Partial regularity for BV minimizers. Arch. Ration. Mech. Anal., June 2019, Volume 232, Issue 3, pp. 1429–1473.

[47] Gmeineder, F.; Raita, B.: On critical L^p-differentiability of BD-maps. Rev. Mat. Iberoam., 35(7), 2019, pp. 2071–2078.

[48] Gobert, J.: Une inéquation fondamentale de la théorie de l'élasticité. Bull. Soc. Roy. Sci. Liege 3-4, 182–191 (1962).

[49] Kirchheim, B.; Kristensen, J., 2016. On rank one convex functions that are homogeneous of degree one. Arch. Ration. Mech. Anal. 221, no. 1, 527-558.

[50] Kohn, R.V.: New Estimates for Deformations in Terms of their Strains. Ph.D. Thesis, Princeton Univ., 1979.

[51] Kohn, R.V.: New integral estimates for deformations in terms of their nonlinear strains. Arch. Rational Mech. Anal. 78 (1982), no. 2, 131–172.

[52] Kolyada, V.I.: On the embedding of Sobolev spaces. (Russian) Mat. Zametki 54 (1993), no. 3, 48–71, 158; translation in Math. Notes 54 (1993), no. 3-4, 908–922 (1994).

[53] Kristensen, J.; Mingione, G.: The singular set of Lipschitzian minima of multiple integrals. Arch. Ration. Mech. Anal. 184 (2007), no. 2, 341–369.

[54] Kristensen, J.; Rindler, F.: Relaxation of signed integral functionals in BV, Calc. Var. Partial Differential Equations 37 (2010), no. 1-2, 29–62.

[55] Kuusi, T.; Mingione, G.: Partial regularity and potentials. Journal de L'École polytechnique - mathématiques 3 (2016) 309–363.

[56] Maz'ya, V.G.; Shaposhnikova, T.: Theory of Sobolev Multipliers, Grundlehren der mathematischen Wissenschaften (307), Springer, 2009.

[57] Mingione, G.: The singular set of solutions to non-differentiable elliptic systems. Arch. Ration. Mech. Anal. 166 (2003), 287–301.

[58] Mingione, G.: Bounds for the singular set of solutions to non linear elliptic systems. Calc. Var. Partial Differ. Equ. 18 (2003), 373–400.

[59] Mingione, G.: Regularity of minima: an invitation to the dark side of the calculus of variations. Appl. Math. 51 (2006), no. 4, 355–426.

[60] Morrey, C.B., Jr.: Multiple integrals in the calculus of variations. Reprint of the 1966 edition [MR0202511]. Classics in Mathematics. Springer-Verlag, Berlin, 2008. x+506 pp.

[61] Mosolov, P.P., Mjasnikov, V.P.: On the correctness of boundary value problems in the mechanics of continuous media. Math. USSR Sbornik 17(2), 257–267 (1972)

[62] Nečas, J.: Sur les normes équivalentes dans $W_k^p(\Omega)$ et sur la coecivité des formes formellement positives. In: Séminaire Equations aus Dérivées Partielles, Les Presses de l'Université de Montréal, pp. 102–128 (1966).

[63] Ornstein, D.: A non-equality for diferential operators in the L_1-norm, Arch. Rational Mech. Anal. 11 (1962), 40-49.

[64] Reshetnyak, Yu. G.: Weak convergence of completely additive vector functions on a set. Sibirsk. Mat. Ž. 9 (1968), 1386– 1394.

[65] Rindler, F.: Lower semicontinuity for integral functionals in the space of functions of bounded deformation via rigidity and Young measures. Arch. Ration. Mech. Anal. 202 (2011), no. 1, 63–113.

[66] Schmidt, T.: Partial regularity for degenerate variational problems and image restoration models in BV. Indiana Univ. Math. J. 63, 213–279 (2014).

[67] Schmidt, T.: BV Minimizers of Variational Integrals: Existence, Uniqueness, Regularity. Habilitationsschrift (postdoctoral thesis), Friedrich-Alexander-Universität Erlangen-Nürnberg, 86 pp. (2015).

[68] Schmidt, T.: A simple partial regularity proof for minimizers of variational integrals. NoDEA Nonlinear Differential Equations Appl. 16 (2009), no. 1, 109–129.

[69] Strang, G.; Temam, R.: Functions of bounded deformation, Arch. Rat. Mech. Anal. 75 (1981), 7–21.

[70] Strauss, M. J.: Variations of Korn's and Sobolev's inequalities, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R.I., 1973, pp. 207-214.

[71] Suquet, P. M.: Un espace fonctionnel pour les ´eequations de la plasticit´e. Ann. Fac. Sci. Toulouse Math. (5), 1(1): 77–87, 1979.

[72] Triebel, H.: Theory of Function Spaces. Leipzig, Geest & Portig 1983, Basel, Birkhäuser 1983.

[73] Van Schaftingen, J.: Limiting Sobolev inequalities for vector fields and canceling linear differential operators. J. Eur. Math. Soc. (JEMS) 15 (2013), no. 3, 877–921.

[74] Zhang, K.: A construction of quasiconvex functions with linear growth at infinity. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), no. 3, 313–326.

Declaration

MATHEMATICAL INSTITUTE, UNIVERSITY OF BONN, ENDENICHER ALLEE 60, 53115 BONN, GERMANY
E-mail address: `fgmeined@math.uni-bonn.de`