FA Revision

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Functional Analysis Revision Class

Problem 1: Terminology

Give examples of

- Banach spaces which are not Hilbert,
- Banach spaces which are not separable,
- Banach spaces on which not any two norms are equivalent,
- a linear operator T on some Banach space X which is bounded, but discontinuous,
- a linear operator $T\colon X \to Y$ between two Banach spaces X, Y which is bounded and continuous,
- a linear operator $T\colon X \to Y$ between two Banach spaces X, Y which is not bounded.

Moreover, decide whether the following are true or false – if they are false, give the correct statement:

- For any $1 \leq p \leq \infty$ and any $f \in \mathrm{L}^p(\mathbb{R}^n)$, there exists $(f_j) \subset \mathrm{C}_c^\infty(\mathbb{R}^n)$ such that $f_j \to f$ in $\mathrm{L}^p(\mathbb{R}^n)$.
- For any $1 \leq p \leq \infty$ we have $\mathrm{L}^p(\Omega)^* \cong \mathrm{L}^{p'}(\Omega)$, where $p' = \frac{p}{p-1}$ if $1 < p < \infty$, $1' = \infty$, $\infty' = 1$.

Problem 2:

Decide with proof which of the following are proper dense subspaces of $\ell^2(\mathbb{N})$:

$$\begin{aligned}
\mathfrak{A} &:= \{x = (x_j) \in \ell^2(\mathbb{N})\colon\ x_{2019} \geq 0\},\\
\mathfrak{B} &:= \{x = (x_j) \in \ell^2(\mathbb{N})\colon\ x_{2019} = 0\},\\
\mathfrak{C} &:= \{x = (x_j) \in \ell^2(\mathbb{N})\colon\ \sum_j |\sin(x_j)| < \infty\}.
\end{aligned}$$

Problem 3:

Decide with proof whether the following subsests of $\ell^2(\mathbb{N})$ are bounded and/or precompact and/or compact:

$$\begin{aligned}
\mathcal{A} &:= \{x = (x_j)\colon\ \|x\|_{\ell^2} \leq 1\},\\
\mathcal{B} &:= \{x = (x_j)\colon\ |x_j| \leq \frac{1}{\sqrt{j}} \text{ for all } j \in \mathbb{N}\},\\
\mathcal{C} &:= \{x = (x_j)\colon\ |x_j| \leq \frac{1}{j} \text{ for all } j \in \mathbb{N}\}.
\end{aligned}$$