Seminar Syllabus:
Inequalities in Analysis & Geometry

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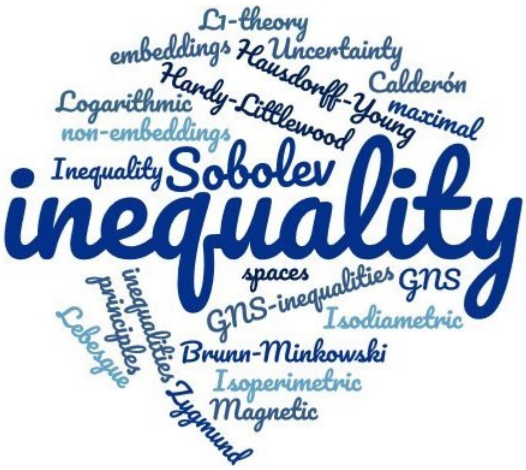

First meeting (Oct 2019), FG

We set the theme of the seminar and recall the underlying basics: Lebesgue spaces, distributions and Sobolev spaces. It is certainly favorable if you already have some preknowledge on these matters, but it is not strictly required.

Talk 1: The Hardy-Littlewood maximal operator and the fractional integration theorem

We introduce the Hardy-Littlewood maximal operator and establish boundedness between Lebesgue spaces $\mathrm{L}^p \to \mathrm{L}^p$, $1 < p < \infty$. As a consequence, we obtain a quick approach to Lebesgue points of L^p-functions. In the second part of the talk, we address how the boundedness of the HL-maximal operator can be utilised to infer boundedness of fractional integration operators

$$\mathcal{I}_s f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-s}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n$$

between Lebesgue spaces.

Talk 2: The Sobolev embedding theorem (GNS inequalities)

In this talk we discuss the Sobolev embedding theorem, giving rise to the following metaprinciple: Additional information on the integrability of the gradient allows to improve the integrability of the functions themselves. We give two approaches to the topic, one by slicing (originally due to Gagliardo and, indepedently, Nirenberg) and one by use of the fractional integration theorem. The latter approach does not directly work for $p = 1$, and in this respect we discuss Maz'ya's truncation trick to infer that the fractional integration approach equally works for $p = 1$.

Talk 3: Variants of GNS: Higher order and logarithmic inequalities

In this talk we address variations of the theme of GNS-inequalities. First, we provide suitable higher order variants of Sobolev inequalities and cover Morrey's inequality. As a main upshot, such higher integrability or continuity results depend on the underlying space dimension n. For instance, the integrability improvement of the foregoing GNS-inequalities *does* depend on n, and the second part of the talk addresses a way of how to obtain dimension independent integrability improvements – this is the logarithmic Sobolev inequality.

Talk 4: Non-Embeddings

The inequalities having been encountered so far only hold for a certain range of exponents p depending on the underlying space dimension n and the geometry of the underlying domain Ω. In this talk we discuss borderline cases of exponent constellations (usually finding that the borderline cases require a completely different treatment) and badly behaved domains on which the desired inequality or embeddings, respectively, fail to hold.

Talk 5: Functions of bounded variation and the isoperimetric inequality

When we extend the Sobolev embedding theorems of the previous talks to functions of bounded variation (a strict superset of all Sobolev spaces $\mathrm{W}^{1,p}(\mathbb{R}^n)$, elements of which being allowed to have jumps), the Sobolev embedding theorem converts into the *isoperimetric inequality*. The talk aims to give a quick and comprehensive overview of BV-functions and then concludes with the derivation of the isoperimetric inequality. This inequality allows to bound the Lebesgue measure of a (sufficiently nice) set against the $(n-1)$-dimensional Hausdorff measure of its boundary.

Talk 6: The isoperimetric and Brunn-Minkowski inequalities

In this talk we aim to find the sharp constant in the isoperimetric inequality. This is based on the geometric Brunn-Minkowski inequality, allowing to estimate $\sqrt[n]{\mathscr{L}^n(A)} + \sqrt[n]{\mathscr{L}^n(B)} \leq \sqrt[n]{\mathscr{L}^n(A+B)}$ for non-empty, compact subsets $A, B \subset \mathbb{R}^n$. We give a proof of the Brunn-Minkowski inequality, and then establish that the optimal constant in the isoperimetric inequality *is the same as the optimal constant in the Sobolev inequality for* BV*-functions*.

Talk 7: Steiner symmetrisation and the isodiametric inequality

Whereas the isoperimetric inequalities allows to infer that among all (sufficiently nice) subsets of a certain perimeter, spheres maximise the Lebesgue measure, the isodiametric inequality deals with this question when the diameter is fixed. The isodiametric inequality is established via the method of Steiner symmetrisation of sets, and this talk aims to give a comprehensive introduction to the theme, leading to a proof of the isodiametric inequality.

Talk 8: Nash's inequality and semigroups

As an outlook, we here present an application to smoothing estimates for certain operator semigroups. These turn out to be equivalent with a different look on the Sobolev inequality due to Nash. In this respect, the talk intends to give a quick and comprehensive overview of (contraction) semigroups (e.g., associated with evolution equations such as the heat equation). In the second part, Nash's inequality is introduced and discussed, finally yielding the requisite smoothing estimates.

Talk 9: Interlude: The diamagnetic inequality

Before we return to the study of Sobolev-type inequalities, we pause and introduce the magnetic Sobolev spaces $H^1_A(\mathbb{R}^n)$. The latter consists of all functions $f\colon \mathbb{R}^n \to \mathbb{C}$ for which $f \in \mathrm{L}^2(\mathbb{R}^n;\mathbb{C})$ and $(\partial_j + \mathrm{i}\, A_j)f \in \mathrm{L}^2(\mathbb{R}^n;\mathbb{C})$ for $j \in \{1,..,n\}$, $A = (A_j)$ being a suitable vector potential. We then briefly discuss the *diamagnetic inequality*, leading to a pointwise estimate of $|\nabla|f||$ by $|(\nabla + \mathrm{i}\, A)f|$. This talk is essentially decoupled from all of the foregoing and afterward talks.

Talk 10: Calderón-Zygmund inequality and singular integrals

The key feature of the fractional integrals of the first talk is that $1/|y|^{n-s}$ has an integrable singularity in $y = 0$ provided $0 < s < n$. If $s = 0$, then the corresponding operator needs to be interpreted differently, and turns out to extend to extend to a bounded linear operator $\mathrm{L}^p \to \mathrm{L}^p$ for if $1 < p < \infty$. The talk is centered around proving this fact, and discussing implications for the regularity of the Poisson (and more generally, elliptic) equations. As a consequence, we obtain a vast class of Korn-type inequalities. Such inequalities allow to bound the L^p-norms of *full* k-th order gradients against those of certain elliptic differential expressions, for $1 < p < \infty$.

Talk 11: Ornstein's Non-Inequality

In the last talk we shall have seen that Korn-type inequalities cannot be inferred from singular integral estimates for $p = 1$. In *this talk*, however, we discover that such estimates simply do not hold in general – manifested by the so-called Ornstein's Non-Inequality. We give a proof of Ornstein's Non-Inequality and discuss its implications – so, for instance, that the space BD of functions of bounded deformation is strictly larger than BV.

Talks 12 and 13: L^1-theory: The general case

We have seen in talk 2 that potential theoretic arguments allow to infer GNS-type inequalities. If we are interested in inequalities $\|\varphi\|_{\mathrm{L}^1} \leq \|A[D]\varphi\|_{\mathrm{L}^1}$, where $A[D]$ is a first order operator, then the techniques and methods presented in talk 2 do *not* extend to this situation. On the other hand, the results of talk 11 rule out to reduce such inequalities ($p = 1$!) to the full gradient setting. This motivates the quest of *what survives.* In these talks, we present the fundaments of the theory developed by Van Schaftingen, Bourgain and Brezis to deal with such inequalities - a topic that continues to be of interest up to date.

Note: Talks 12 & 13 require some knowledge about elementary commutative algebra and rudimentary algebraic geometry such as the Hilbert Nullstellensatz. If you wish to give one of these talks, make sure to briefly recap the underlying essentials.