

Regularity Theory

Research Overview

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1 Introduction

One part of my research is devoted to the regularity for (mainly elliptic variational) problems. Here we consider variational integrals

$$\mathcal{F}_f[u] := \int_{\Omega} F(\nabla u) \, dx, \quad \mathcal{F}_s[u] := \int_{\Omega} F(\varepsilon(u)) \, dx$$

with $\Omega \subset \mathbb{R}^n$ an open Lipschitz domain and $F: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ an integrand of *linear growth*. As regards \mathcal{F}_s , $\varepsilon(u) := \frac{1}{2}(Du + Du^\top)$ denotes the symmetric gradient of a map $u: \Omega \rightarrow \mathbb{R}^n$. Linear growth means that $c_1|z| \leq F(z) \leq c_2(1 + |z|)$ for all $z \in \mathbb{R}^{N \times n}$, where $N = n$ for \mathcal{F}_s . A prominent example here is the *area integrand* $F := \sqrt{1 + |\cdot|^2}$. By this growth assumption, the direct method of the Calculus of Variations reveals compactness issues for proving weak convergence of minimising sequences; the (symmetric) gradients of such sequences might concentrate, for instance. Thus one first relaxes the corresponding variational problems to the space $BV(\Omega; \mathbb{R}^N)$ in the full gradient case and to $BD(\Omega)$ in the symmetric gradient case. The latter space is defined as the linear space of all $v \in L^1(\Omega; \mathbb{R}^n)$ such that $\varepsilon(v)$ is a finite $\mathbb{R}^{n \times n}$ -valued Radon measure. By Ornstein's Non-Inequality (that is, the failure of the Calderón-Zygmund theory in the L^1 -setting), generic maps from BD need not even possess full distributional derivatives in L^1_{loc} or the $\mathbb{R}^{n \times n}$ -valued Radon measures. Thus one seeks conditions on F to produce minima of the relaxed functionals *which are more regular than generic competitor maps from BV or BD* , respectively. Here, questions different from the usual Sobolev space $W^{1,p}$ -setup, $1 < p < \infty$, emerge; for a comprehensive account of the latter, we refer to MINGIONE [Mi06]. So, for instance, even the proof of $W^{1,1}$ -regularity of minima is in general far from obvious.

2 Selected Results

The following sections present selected results obtained partially in collaboration with others, usually being at the interface of harmonic analysis, function spaces and elliptic regularity theory.

2.1 Full gradient functionals, convex case

It is known from the foundational work of BILDHAUER [Bi02] and, more recently BECK & SCHMIDT [BS13], that minimisers of relaxed Dirichlet problems on BV are $W^{1,1}$ -regular *provided the integrands F feature an ellipticity which is at most as degenerate as that of the area integrand*. This amounts to saying that no $W^{1,1}$ -regularity were available for integrands which are a -elliptic, $a > 3$.

Jointly with L. BECK (Augsburg) and M. BULÍČEK [1], we established the first *unconditional* $W^{1,1}$ -regularity of minima of the relaxed formulations of Neumann-type problems on BV . Here, *unconditional* refers to the fact that the integrands are

allowed to be as degenerate as possible, as long they are convex and have positive definite Hessian everywhere. Rather than employing difference quotients estimates, the possibly very strong degeneration of the integrands' ellipticities do not give higher integrability of the minimisers' gradients *but their absolute continuity for \mathcal{L}^n* . The proof crucially relies on Chacon's biting lemma, and revolves around showing that the biting limit of a suitable minimising sequence is curl-free. Thus, assuming that Ω is simply connected, it arises as the gradient of some $W^{1,1}$ -map, which eventually yields the desired vanishing of the singular part.

2.2 Symmetric gradient functionals, convex case

For Dirichlet problems related to functionals \mathcal{F}_s , common strategies that rely on difference quotients are bound to cause issues. This is so because of Ornstein's Non-Inequality and the fact that minimisers are a priori not known to possess full derivatives in L^1 (or in the Radon measures). However, as we discovered in collaboration with J. KRISTENSEN (Oxford) [7] partially based on [3, 2], Ornstein's Non-Inequality is invisible on the level of fractional Sobolev spaces. This enabled us to obtain suitable estimates in Besov-Nikolskiĭ spaces for a good range of degenerate ellipticities of F . Most instrumentally, by uniqueness issues in the linear growth setup, though, this result concerns *all* minimisers of the relaxed problems. For *one* particular generalised minimiser, this result can be actually extended to yield *exactly the same Sobolev regularity and partial Hölder results* in the convex case as is presently known for BV [5]. Compared with the results outlined for symmetric-quasiconvex functionals below, this particular partial regularity result is of independent interest as it is able to cope with (essentially) all sorts of possible degeneration of the integrands as long as *some* ellipticity is preserved.

2.3 Partial regularity in the quasiconvex case

On the other hand, the variational problems associated with \mathcal{F}_f (or \mathcal{F}_s) are *vectorial* in the sense that competitors are vector fields. In this framework, it is well-known that minima in general are not *fully* $C^{1,\alpha}$ -regular but only *partially regular*. This means that minimisers are $C^{1,\alpha}$ in the neighbourhoods of each element of a set which is open and whose complement has \mathcal{L}^n -measure zero – the singular set.

Since it is often equivalent to sequential weak(*) lower semicontinuity of variational integrals, MORREY's notion of *quasiconvexity* has a special standing in the vectorial Calculus of Variations. Starting with EVANS' seminal paper [Ev86] on the partial regularity of minima of quasiconvex variational integrals in the superquadratic ($p \geq 2$) setup, the partial regularity of minima in the linear growth case *had been open ever since*. Jointly with J. KRISTENSEN (Oxford) [8], we established the first partial regularity result in the quasiconvex, linear growth situation. The proof, which is based on a direct comparison principle for minima on *good* balls, is entirely direct and shall also turn out useful for other situations [9]. In particular, even though it requires substantial modification, a variant of the proof can also be made work in the case of symmetric-quasiconvex (i.e., quasiconvex on the symmetric matrices) linear growth functionals, [2, 6]. In view of comparison problems, this requires novel Fubini type theorems for BD; other than for BV-maps, the tangential derivatives of BD-maps in general do not exist in L^1 even on a small, but non-empty set of 'good' spheres. Among others, it is here where we combine harmonic analysis, function spaces and elliptic regularity, and connections to more general differential operators will be explored in the future; for this, see the research exposé on L^1 -estimates.

References

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