Functional Analysis & PDEs

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Functional Analysis Revision

Give yourself 120–140 minutes to work on the following five problems.

Problem 1: **10 marks**

Prove or disprove whether

$$\{x = (x_j) \in \ell^1(\mathbb{N}) \colon \sum_j x_j = 0\}$$

is dense in $(\ell^2(\mathbb{N}), \|\cdot\|_{\ell^2(\mathbb{N})})$.

Problem 2: **4 + 6 = 10 marks**

Let $1 \leq p \leq q \leq \infty$.

(a) Prove that $\ell^p(\mathbb{N}) \subset \ell^q(\mathbb{N})$.

(b) Let $T \colon \ell^p(\mathbb{N}) \ni x \mapsto x \in \ell^q(\mathbb{N})$ be the injection underlying (a). Show that T is a bounded linear operator and compute its operator norm.

Problem 3: **10 marks**

Let $1 \leq q \leq \infty$. Prove that

$$\bigcup_{p<q} \ell^p(\mathbb{N}) \subsetneq \ell^q(\mathbb{N}).$$

Problem 4: **10 marks**

Let $\mathrm{C}([0,1])$ the space of continuous functions $u \colon [0,1] \to \mathbb{R}$, endowed with the usual supremum norm. Let $X \subset \mathrm{C}([0,1])$ be a closed subspace of $\mathrm{C}([0,1])$ for the supremum norm which satisfies

$$X \subset \bigcup_{0<\alpha\leq 1} \mathrm{C}^{0,\alpha}([0,1]).$$

Prove that $\dim(X) < \infty$.

Problem 5: **10 marks**

Let $\mathcal{H}$ be separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. Let (e_j) be an orthonormal basis for $\mathcal{H}$ and let (x_n) be a sequence in $\mathcal{H}$. Prove that the following are equivalent:

(a) For all $f \in \mathcal{H}^*$ there holds $f(x_n) \to 0$ as $n \to \infty$.

(b) For all $j \in \mathbb{N}$ there holds $\langle e_j, x_n \rangle \to 0$ as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$.