$\mathscr{A}$-quasiconvexity, function spaces & regularity

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partly based on joint work with

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Universität Konstanz

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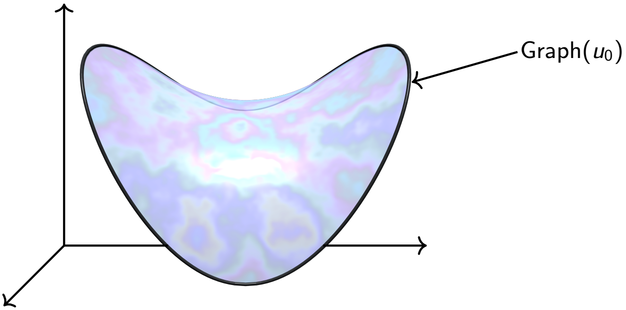
Minimal Surfaces

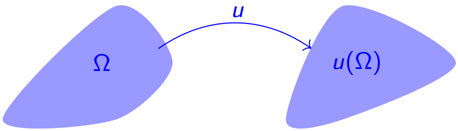
The classical minimal surface problem reads as

$$\text{minimise} \quad \mathscr{F}[u] := \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x \quad \text{subject to} \quad u|_{\partial\Omega} = u_0.$$

- Graphs of minimisers yield *minimal surface* with 'boundary datum' u_0.



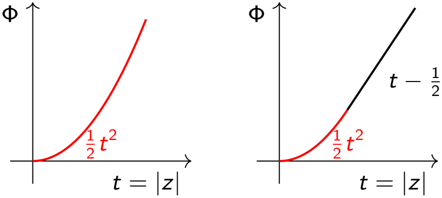
Elasticity and plasticity



- Symmetric gradient: $\varepsilon(u) := \frac{1}{2}(Du + Du^{\top})$
- Trace-free symmetric gradient: $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\operatorname{div}(u)E_n$

$$\text{minimise} \quad \mathscr{F}[u] := \int_\Omega \Phi(|\varepsilon^D(u)|)\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\operatorname{div}(u)|^2\,\mathrm{d}x - \int_\Omega F \cdot u\,\mathrm{d}x$$

subject to suitable side constraints (forces, tensions)



A unifying framework I

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with $\mathbb{A}_\alpha \in \mathscr{L}(V;W)$,

$$\mathbb{A} = \sum_{|\alpha|=k} \mathbb{A}_\alpha \partial^\alpha \colon v \mapsto \sum_{|\alpha|=k} \mathbb{A}_\alpha \partial^\alpha v, \qquad v\colon \Omega \to V$$

a vectorial differential operator. We aim to minimise

$$\mathscr{F}[u] := \int_\Omega f(\mathbb{A}u)\,\mathrm{d}x \qquad \text{over suitable maps } u \text{ with } u|_{\partial\Omega} = u_0,$$

where $u_0\colon \Omega \to V$ is a suitable Dirichlet datum and f **has** $1 \leq p < \infty$ **growth**:

$$|f(z)| \leq c(1+|z|^p) \qquad \text{for all } z \in W.$$

Example (The symmetric gradient)

$V=\mathbb{R}^n$, $W=\mathbb{R}^{n\times n}_{\mathrm{sym}}$, $\mathbb{A}u := \varepsilon(u) := \frac{1}{2}(Du + Du^\top)$

Example (The trace-free symmetric gradient)

$V=\mathbb{R}^n$, $W=\mathbb{R}^{n\times n}_{\mathrm{tf,sym}}$, $\mathbb{A}u := \varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\mathrm{div}(u)E_n$

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A unifying framework II

$$\mathscr{F}[u] := \int_\Omega f(\mathbb{A}u)\,\mathrm{d}x, \quad |f(z)| \leq c(1+|z|^p)$$

1. $|\mathbb{A}u|^p$ should be integrable and u should attain the right boundary values $\rightsquigarrow$ denote this class $\mathcal{X}_p$.
2. (v_i) in $\mathcal{X}_p$ with $\mathscr{F}[v_i] \to \inf_{\mathcal{X}_p} \mathscr{F}$
3. Hope for boundedness of (v_i) in $\mathcal{X}_p$ to extract a suitably convergent subsequence in a weak sense: $v_{i(j)} \rightsquigarrow v$
4. Sequential LSC for '$\rightsquigarrow$': $\mathscr{F}[v] \leq \liminf_{j\to\infty} \mathscr{F}[v_{i(j)}]$

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Reasonable:

$$\mathrm{W}^{\mathbb{A},p}(\Omega) := \{u\colon\ \|u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)} < \infty\}$$

Natural question: When do we have

$$\mathrm{W}^{\mathbb{A},p}(\Omega) \simeq \mathrm{W}^{k,p}(\Omega; V)?$$

Depends on $\mathbb{A}$ and whether $1 < p < \infty$ or $p = 1$!

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Requires a generalisation of the

Quasiconvexity á la Morrey: $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ continuous is called **quasiconvex** if

$$f(z) \leq \int_{(0,1)^n} f(z+\nabla\varphi)\,\mathrm{d}x \qquad \forall z \in \mathbb{R}^{N\times n},\ \varphi \in \mathrm{C}_c^\infty((0,1)^n;\mathbb{R}^N).$$

$\longrightarrow$ will lead us to $\mathscr{A}$-quasiconvexity a la Fonseca & Müller

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$\Downarrow$

Existence of minima

$\Downarrow$

What can we say about their regularity?

Plan of the talk

Function spaces, harmonic analysis
and boundary behaviour of maps

$\downarrow$

Coercivity and LSC:
p-strong $\mathscr{A}$-quasiconvexity

$\downarrow$

$\mathrm{C}^{1,\alpha}$-partial regularity of minimisers

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$\mathbb{A}$ versus ∇^k – Calderón-Zygmund & Ornstein

Theorem (Korn versus Ornstein)

In general, the inequality $\|\nabla^k u\|_{\mathrm{L}^p} \leq c\|\mathbb{A}u\|_{\mathrm{L}^p}$ holds for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n;V)$ if and only if $\mathbb{A}$ is elliptic and $1 < p < \infty$ – but not for $p = 1$.

- We call $\mathbb{A}$ **elliptic** $\Leftrightarrow$

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}: \quad \mathbb{A}[\xi] := \textstyle\sum_{|\alpha|=k} \xi^\alpha \mathbb{A}_\alpha : V \to W \text{ is injective}$$

- Write for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n;V)$ and $|\alpha| = k$:

$$\partial^\alpha u = c_n \mathscr{F}^{-1}\Big(\underbrace{\xi^\alpha (\mathbb{A}[\xi]^* \mathbb{A}[\xi])^{-1} \mathbb{A}[\xi]^*}_{=m_\alpha(\xi)} \mathscr{F}[\mathbb{A}u]\Big)$$

$\longrightarrow$ m_α belongs to $\mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}; \mathscr{L}(W;V))$ and is homogeneous of degree zero.

$\longrightarrow$ Apply **Theorem of Mihlin-Hörmander/Calderón-Zygmund**.

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What survives? – Strengthening ellipticity

- **Sobolev inequality:**
  Van Schaftingen (JEMS, '13), based on Bourgain & Brezis (JAMS, '07):

$$\|u\|_{\mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n)} \lesssim \|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n)} \text{ for } u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$$

$$\iff \mathbb{A} \text{ elliptic and } \bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \mathbb{A}[\xi](V) = \{0\} \text{ (cancelling)}$$

- **Trace inequalities?**

Example (The trace-free symmetric gradient)

The operator $\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\operatorname{div}(u)E_n$

- $n = 2$: elliptic, but:

$$\varepsilon^D(u) \overset{!}{=} 0 \implies \begin{cases} \partial_1 u_1 & = \partial_2 u_2 \\ \partial_2 u_1 & = -\partial_1 u_2 \end{cases} \quad \text{Cauchy-Riemann!}$$

$f : \mathbb{D} \ni z \mapsto \frac{1}{z-1} \in \mathbb{C}$ holomorphic and $\int_{\partial\mathbb{D}} |f| \, \mathrm{d}\mathscr{H}^1 = +\infty$.

- $n \geq 3$: $\ker(\varepsilon^D) = \{\text{Killing fields}\} \subset \mathscr{P}_2(\mathbb{R}^n; \mathbb{R}^n)$

Traces and boundary behaviour

Theorem (Breit, Diening, FXG, APDE '20 + Diening & FXG '21)

The following are equivalent for a k-th order operator $\mathbb{A}$ and $1 \leq p < \infty$:

1. $\mathbb{A}$ *is* $\mathbb{C}$*-elliptic, so*

$$\mathbb{A}[\xi]\colon V + \mathrm{i}V \to W + \mathrm{i}W \qquad \text{is injective for all } \xi \in \mathbb{C}^n \setminus \{0\}.$$

2. *For all open, bounded and smooth $\Omega \subset \mathbb{R}^n$ there holds*

$$\mathrm{Tr}_{\partial\Omega}(\mathrm{W}^{k,p}(\Omega;V)) = \mathrm{Tr}_{\partial\Omega}(\mathrm{W}^{\mathbb{A},p}(\Omega))$$

- hinges on the Hilbert Nullstellensatz from algebraic geometry

$$u(x) = \Pi u(x) + \int_{\mathrm{B}} K_{\mathbb{A}}(x-y)\mathbb{A}u(y)\,\mathrm{d}y \quad \Rightarrow \quad \ker(\mathbb{A}) \subset \mathscr{P}_m(\mathbb{R}^n;\mathbb{R}^N)$$

- $\mathbb{C}$-ellipticity is equivalent to $\ker(\mathbb{A})$ being a

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Trace Inequalities & $\mathbb{C}$-ellipticity

$\mathbb{A}$ not $\mathbb{C}$-elliptic

$\downarrow$

$\mathbb{A}$ contains a copy of the two-dimensional ε^D

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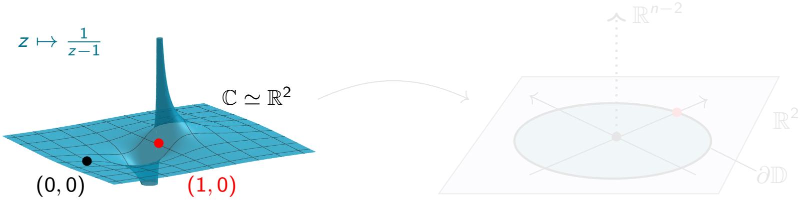
shift singularity in $\mathbb{C} \simeq \mathbb{R}^2$ along $\{0\} \times \mathbb{R}^{n-2}$



Obtain $u \in \mathrm{W}^{\mathbb{A},p}(\mathbb{D} \times (-1,1)^{n-2})$ with $\int_{\partial\mathbb{D}(0,1)\times(-1,1)^{n-2}} |u| \, \mathrm{d}\mathscr{H}^{n-1} = +\infty$

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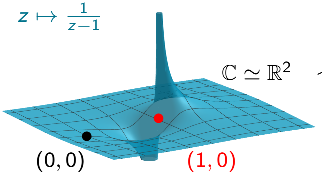

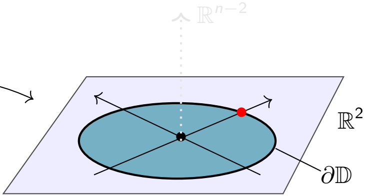

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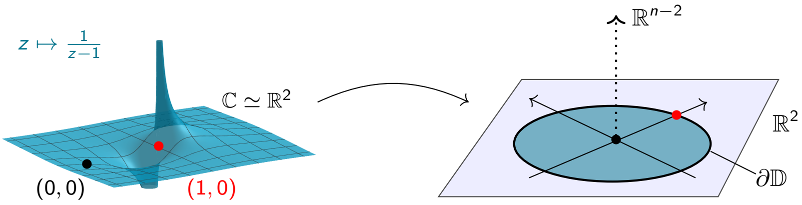

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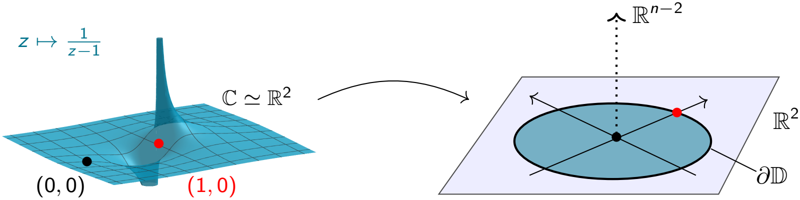

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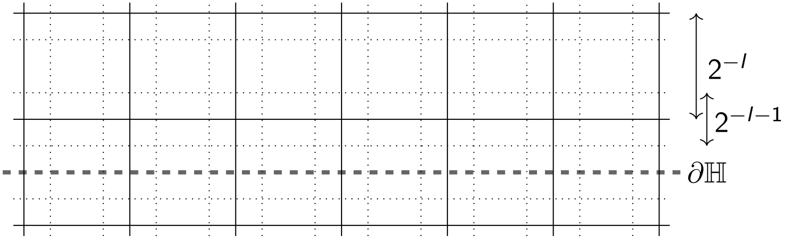
Proof sketch for the halfspace $\mathbb{H} = \{x_n > 0\}$



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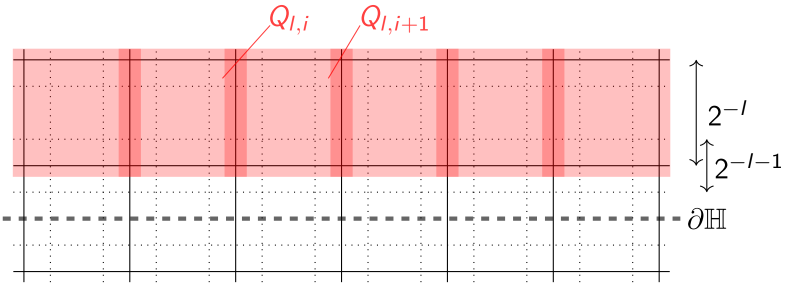

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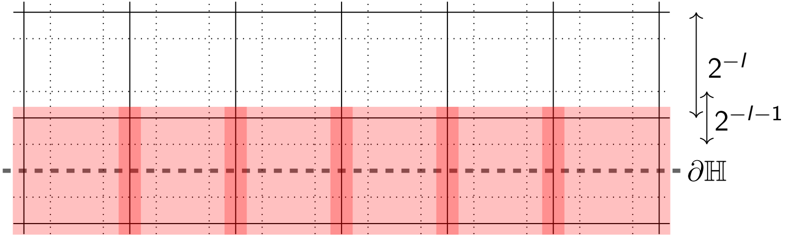

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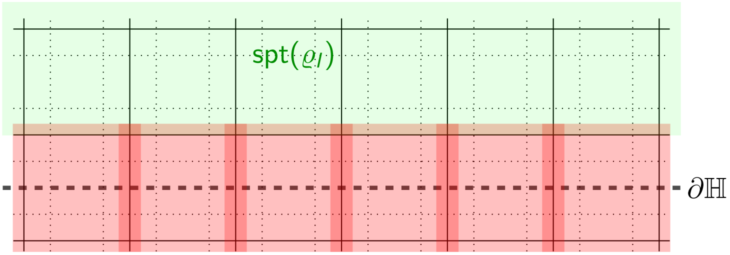

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$$T_l u := \varrho_l u + (1-\varrho_l)\sum_{i\in\mathbb{N}} \rho_{l,i}\Pi_{l,i}u$$

- **Idea**: $u_j \to u$ in $\mathrm{W}^{\mathbb{A},1}(\mathbb{H})$ und $\mathrm{Tr}(u) = \lim_{j\to\infty} \mathrm{Tr}(T_j u)$ in $\mathrm{Tr}(\mathrm{W}^{k,1})(\partial\mathbb{H})$

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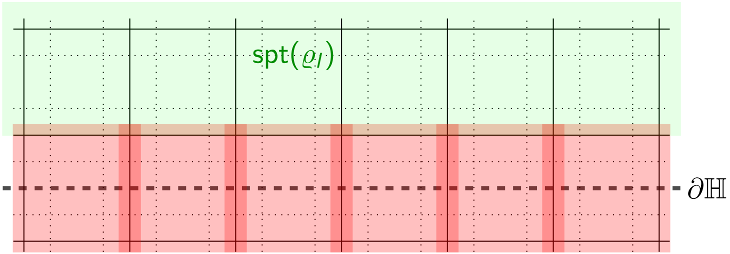

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Proof sketch for the halfspace $\mathbb{H} = \{x_n > 0\}$

- Use $\mathrm{Tr}(u) = \lim_{l\to\infty} \mathrm{Tr}(T_l u) = \sum_{l=-\infty}^{\infty} \mathrm{Tr}(T_{l+1}u - T_l u)$
- Then

$$T_{l+1}u - T_l u = (\varrho_l - \varrho_{l+1})(\dots) + \sum_{i,m\in\mathbb{N}} \varrho_{l+1}\rho_{l,m}\rho_{l+1,i}\ \underbrace{(\Pi_{l+1,i}u - \Pi_{l,m}u)}_{\text{polynomials of fixed degree!}}$$

- Crucial: If $|\alpha| \leq k$,

$$\begin{aligned}\int_{Q_{l,m}} |\partial^\alpha(\Pi_{l+1,i}u - \Pi_{l,m}u)|\,\mathrm{d}x &\lesssim \frac{1}{\ell(Q_{l,m})^{|\alpha|}} \int_{Q_{l,m}} |\Pi_{l+1,i}u - \Pi_{l,m}u|\,\mathrm{d}x \\ &\lesssim \frac{\ell(Q_{l,m})^k}{\ell(Q_{l,m})^{|\alpha|}} \int_{\text{cubes touching } Q_{l,m}} |\mathbb{A}u|\end{aligned}$$

and for $|\beta| = k - |\alpha|$,

$$|\partial^\beta(\varrho_{l+1}\rho_{l,m}\rho_{l+1,i})| \lesssim \frac{1}{\ell(Q_{l,m})^{k-|\alpha|}} \qquad \square$$

Traces and potentials

If $\mathbb{A}$ is a first order differential operator, then in particular

- $\mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^{\frac{n}{n-1}}(\mathbb{R}^n; V)$ if $\mathbb{A}$ is $\mathbb{R}$-elliptic and cancelling,
- $\mathrm{Tr}_{\mathbb{R}^{n-1}\times\{0\}}\colon\ \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n) \hookrightarrow \mathrm{L}^1(\mathbb{R}^{n-1}\times\{0\}; V)$ if $\mathbb{A}$ is $\mathbb{C}$-elliptic.

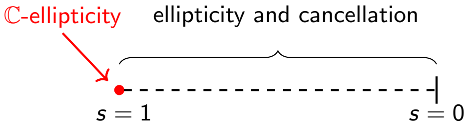
In between:

Theorem (FXG, Raita & Van Schaftingen, Indiana '21)

If $0 \leq s < 1$ and $\Sigma \subset \mathbb{R}^n$ $(n-s)$-dimensional, then

$$\exists\, \mathrm{Tr}_\Sigma\colon\ \mathrm{BV}^{\mathbb{A}}(\mathbb{R}^n) \to \mathrm{L}^{\frac{n-s}{n-1}}(\mathbb{R}^n; \mathscr{H}^{n-s} \llcorner \Sigma)$$

provided $\mathbb{A}$ is $\mathbb{R}$-elliptic and cancelling ($\bigcap_{\xi\neq 0}\mathbb{A}[\xi](V) = \{0\}$).



Where we are now

Function spaces, harmonic analysis
and boundary behaviour of maps

$\downarrow$

$\boxed{\text{Coercivity and LSC: } p\text{-strong } \mathscr{A}\text{-quasiconvexity}}$

$\downarrow$

$C^{1,\alpha}$-partial regularity of minimisers

The historical development

- **Idea:** $Q = (0,1)^n$, $T : Q \to \mathbb{R}^{N\times n}$ and $\mathrm{curl}(T) = 0 \Rightarrow T = \nabla u$.
- We say that $\mathscr{A}$ is an **annihilator** for $\mathbb{A}$, and $\mathbb{A}$ is a **potential** for $\mathscr{A}$ if

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathscr{A}[\xi]} Z \text{ is exact for any } \xi \in \mathbb{R}^n \setminus \{0\}.$$

- Based on Dacorogna (80s), Fonseca & Müller defined

$\mathscr{A}$-quasiconvexity

An integrand $F\colon W \to \mathbb{R}$ is called $\mathscr{A}$-quasiconvex provided

$$F(z) \leq \int_{(0,1)^n} F(z+\psi)\,\mathrm{d}x$$

holds for all $z \in W$, $\psi \in \mathrm{C}^\infty(\mathbb{T}^n; W)$ with $(\psi)_{(0,1)^n} = 0$ and $\mathscr{A}\psi = 0$.

Lower semicontinuity

Call $\mathscr{A}$ a **constant-rank operator** provided $\dim(\mathscr{A}[\xi](W))$ does not depend on $\xi \in \mathbb{R}^n \setminus \{0\}$.

Metatheorem a lá Fonseca & Müller SIAM '99, $1 < p < \infty$

If F is $\mathscr{A}$-quasiconvex and of p-growth, the associated integral functional

$$v \mapsto \int_\Omega F(v)\,\mathrm{d}x$$

is weakly lower semicontinuous along sequences (v_j) with $\mathscr{A} v_j = 0$.

- Paradigm shift:

Theorem (Raita, Calc Var PDE '19)

Any constant rank operator $\mathscr{A}$ has a potential $\mathbb{A}$, and then F is $\mathscr{A}$-QC iff

$$F(z) \leq \int_{(0,1)^n} F(z + \mathbb{A}\varphi)\,\mathrm{d}x \qquad \forall z \in W\, \forall \varphi \in \mathrm{C}_c^\infty((0,1)^n; V).$$

- also see Arroyo-Rabasa & Simental '21: Homological approach

$\longrightarrow$ **Existence of minimisers!**

Where we are now

Function spaces, harmonic analysis
and boundary behaviour of maps

$\downarrow$

Coercivity and LSC:
p-strong $\mathscr{A}$-quasiconvexity

$\downarrow$

$\boxed{\mathrm{C}^{1,\alpha}\text{-partial regularity of minimisers}}$

Partial regularity – main theorem

Theorem (Conti & FXG, '21 based on FXG, J. Math. Pures Appl. '21)

Let $\mathbb{A}$ be an elliptic differential operator of order one and $F\colon W \to \mathbb{R}$ satisfy

(H1) $F \in \mathrm{C}^2(W)$,

(H2) $|F(z)| \lesssim c(1+|z|^p)$ *for all* $z \in W$ *(growth bound,* $1 < p < \infty$*),*

(H3) $F - \ell V_p$ *is* $\mathscr{A}$*-QC.*

Then any local minimiser of the integral functional

$$v \mapsto \int F(\mathbb{A}v)\,\mathrm{d}x$$

is $\mathrm{C}^{1,\alpha}$*-partially regular.*

- higher order equally possible, here first order for simplicity
- Classical setting: – among others Evans, Acerbi, Fusco, Pasarelli di Napoli, Carozza, Mingione, Kristensen, Duzaar, Schmidt, Diening, Fuchs, Breit, ... and **many**, **many** others

Proof outline

The essential cone and span of $\mathbb{A}$

For a differential operator $\mathbb{A}$, define

$$v \otimes_{\mathbb{A}} \xi := \sum_{j=1}^{n} \xi_j \mathbb{A}_j v, \qquad v \in V,\ \xi \in \mathbb{R}^n.$$

We then define the

- **essential cone** by $\mathscr{C}(\mathbb{A}) := \{v \otimes_{\mathbb{A}} \xi\colon\ v \in V, \xi \in \mathbb{R}^n\}$.
- **essential span** by $\mathscr{R}(\mathbb{A}) := \mathrm{span}(\mathscr{C}(\mathbb{A})) \subset W$.

Upshot: If $N := \dim(V)$, then $\mathscr{R}(\mathbb{A}) \hookrightarrow \mathbb{R}^{N\times n}$.

$\longrightarrow$ upon identification, we may assume that $W = \mathscr{R}(\mathbb{A}) \subset \mathbb{R}^{N\times n}$.

For $F\colon W \to \mathbb{R}$ $\mathscr{A}$-quasiconvex, now define

$$G(z) := F(\Pi_{\mathbb{A}}(z)), \qquad z \in \mathbb{R}^{N\times n},$$

with $\Pi_{\mathbb{A}}\colon \mathbb{R}^{N\times n} \to \mathscr{R}(\mathbb{A})$ such that $\Pi_{\mathbb{A}}[\nabla v] = \mathbb{A}v$.

The case $p \geq 2$: Properties of $G = F \circ \Pi_{\mathbb{A}}$

(H1') $G \in C^2$ if $F \in C^2$.

(H2') $|G(z)| \lesssim (1 + |z|^p)$ since F satisfies this estimate.

(H3') As a consequence of the p-strong $\mathscr{A}$-quasiconvexity, with $Q = (0,1)^n$,

$$\nu \int_Q (1 + |z|^2 + |\mathbb{A}\varphi|^2)^{\frac{p-2}{2}} |\mathbb{A}\varphi|^2 \,\mathrm{d}x \leq \int_Q F(z + \mathbb{A}\varphi) - F(z) \,\mathrm{d}x.$$

Thus with $\phi(t) := t^2 + t^p$,

$$\int_Q |D\varphi|^2 + |D\varphi|^p \,\mathrm{d}x \lesssim \int_Q |\mathbb{A}\varphi|^2 + |\mathbb{A}\varphi|^p \,\mathrm{d}x$$

$$\lesssim \int_Q F(\Pi_{\mathbb{A}}(z) + \mathbb{A}\varphi) - F(\Pi_{\mathbb{A}}(z)) \,\mathrm{d}x \lesssim \int_Q G(z + D\varphi) - G(z) \,\mathrm{d}x$$

A note on $1 < p < 2$

More intricate, hinges on Diening's shifted ϕ-functions and

$$\int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \,\mathrm{d}x \lesssim \int_Q \underbrace{(1 + |\Pi_{\mathbb{A}}(z)|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2}_{\simeq \phi_{|\Pi_{\mathbb{A}}(z)|}(D\varphi)} \,\mathrm{d}x$$

The case $p \geq 2$: Properties of $G = F \circ \Pi_{\mathbb{A}}$

- (H1') $G \in C^2$ if $F \in C^2$.
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Thus with $\phi(t) := t^2 + t^p$,

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$$\lesssim \int_Q F(\Pi_{\mathbb{A}}(z) + \mathbb{A}\varphi) - F(\Pi_{\mathbb{A}}(z)) \, dx \lesssim \int_Q G(z + D\varphi) - G(z) \, dx$$

A note on $1 < p < 2$

More intricate, hinges on Diening's shifted ϕ-functions and

$$\int_Q (1 + |z|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2 \, dx \lesssim \int_Q \underbrace{(1 + |\Pi_{\mathbb{A}}(z)|^2 + |D\varphi|^2)^{\frac{p-2}{2}} |D\varphi|^2}_{\approx \phi_{|\Pi_{\mathbb{A}}(z)|}(D\varphi)} \, dx$$

Thank you for your attention!

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