

Partial Regularity for Linear Growth Functionals

based on joint work with Jan Kristensen

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New Trends in the Regularity Theory for PDEs

Linear Growth Functionals – Full Gradient Case

- *Linear growth integrands* are generalisations of the area-integrand

$$z \mapsto \sqrt{1 + |z|^2}$$

- Precisely, for such integrands F there exist $c_1, c_2, c_3 > 0$ such that

$$c_1|z| - c_2 \leq F(z) \leq c_3(1 + |z|) \quad \text{for all } z \in \mathbb{R}^{N \times n}.$$

For some suitable Dirichlet datum $u_0 \in W^{1,1}(\Omega; \mathbb{R}^N)$, minimise

$$\mathcal{F}[u] := \int_{\Omega} F(Du) \, dx \quad \text{over a Dirichlet class } W_{u_0}^{1,1}(\Omega; \mathbb{R}^N).$$

- Compactness considerations → let

$$BV(\Omega; \mathbb{R}^N) := \{u \in L^1(\Omega; \mathbb{R}^N) : Du \in \mathcal{M}(\Omega; \mathbb{R}^{N \times n})\}$$

→ Even though \mathcal{F} is well-defined on $W^{1,1}$, what is $\mathcal{F}[u]$ for $u \in BV$?

Relaxed Formulation

L. AMBROSIO & G. DAL MASO '93, I. FONSECA & S. MÜLLER '94:

If $F: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ is quasiconvex, then the Lebesgue-Serrin extension of \mathcal{F}

$$\overline{\mathcal{F}}[u] := \inf \left\{ \liminf_{k \rightarrow \infty} \int_{\Omega} F(\nabla u_k) \, dx : \begin{array}{l} (u_k) \subset W_{u_0}^{1,1}(\Omega; \mathbb{R}^N) \\ u_k \xrightarrow{*} u \text{ in } BV(\Omega; \mathbb{R}^N) \end{array} \right\}$$

is given for $u \in BV(\Omega; \mathbb{R}^N)$

$$\begin{aligned} \overline{\mathcal{F}}[u] = & \int_{\Omega} F(\nabla u) \, dx + \int_{\Omega} F^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| \\ & + \int_{\partial\Omega} F^\infty(\operatorname{Tr}(u_0 - u) \otimes \nu_{\partial\Omega}) \, d\mathcal{H}^{n-1}. \end{aligned}$$

Here, $F^\infty(z) := \limsup_{t \searrow 0} tF(z/t)$ is the recession function of F .

Some History – Selected Partial Regularity Results

- ✓ L.C. EVANS '86, ACERBI & FUSCO '87:
Superquadratic QC case ($p \geq 2$)
- ✓ G. ANZELLOTTI & M. GIAQUINTA '88:
 $1 \leq p < \infty$ + convexity
- ✓ M. CAROZZA & A. PASSARELLI DI NAPOLI '97:
Subquadratic QC case ($\frac{2n}{n+2} < p < 2$)
- ✓ M. CAROZZA, N. FUSCO, G. MINGIONE '98:
Subquadratic QC case ($1 < p < 2$)
- ✓ L. DIENING, D. LENGELE, B. STROFFOLINI, A. VERDE '12:
Orlicz growth with $\varphi \in \Delta_2 \cap \nabla_2$, QC

... and many others

→ Linear Growth Functionals on BV?

Generalised Minima

Generalised Minima/ BV-minima

Let $F: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$ be a quasiconvex integrand with linear growth. A map $u \in BV(\Omega; \mathbb{R}^N)$ is called a *generalised minimiser/ BV minimiser* for \mathcal{F} subject to u_0 if and only if

$$\overline{\mathcal{F}}[u] \leq \overline{\mathcal{F}}[v] \quad \text{for all } v \in BV(\Omega; \mathbb{R}^N).$$

- Generalised minima always exist.
- Regularity results available so far **only in the convex context**:
 - ANZELLOTTI & GIAQUINTA '86: Partial Regularity
 - BECK & SCHMIDT '13: $W^{1,1}$ -regularity
 - Precursors by SEREGIN '90s and BILDHAUER '02
→ More in Lisa's talk tomorrow!

→ QC is the central notion for existence of minima - yet no regularity results are available in this case!

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From full to symmetric gradients – Functionals on BD

- Now let $F: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ be of **linear growth**:

$$|z| - 1 \lesssim F(z) \lesssim 1 + |z| \quad \text{for all } z \in \mathbb{R}_{\text{sym}}^{n \times n}.$$

- For some suitable Dirichlet datum u_0 , consider the minimisation of

$$\mathcal{F}[u] := \int_{\Omega} F(\varepsilon(u)) \, dx \quad \text{over a suitable Dirichlet class}$$

- Coercive- and compactness considerations \longrightarrow let

$$\text{BD}(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^n) : \mathbb{E}u \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})\}$$

Ornstein's Non-Inequality $\implies \text{BV}(\Omega; \mathbb{R}^n) \subsetneq \text{BD}(\Omega)$.

- $u \in \text{BD}(\Omega)$ **generalised minimiser** : $\Leftrightarrow \overline{\mathcal{F}}[u] \leq \overline{\mathcal{F}}[v] \ \forall v \in \text{BD}(\Omega)$

$$\begin{aligned} \overline{\mathcal{F}}[u] := & \int_{\Omega} F(\mathcal{E}u) \, d\mathcal{L}^n + \int_{\Omega} F^\infty\left(\frac{d\mathbb{E}^s u}{d|\mathbb{E}^s u|}\right) d|\mathbb{E}^s u| \\ & + \int_{\partial\Omega} F^\infty(\text{Tr}(u_0 - u) \odot \nu_{\partial\Omega}) \, d\mathcal{H}^{n-1} \end{aligned}$$

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Lower Semicontinuity & Notions of Convexity

- $F: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ **symmetric quasiconvex**

$\iff F(A) \leq \int_Q F(A + \varepsilon(\varphi)) \, dx \quad \text{for all } A \in \mathbb{R}_{\text{sym}}^{n \times n} \text{ and } \varphi \in W_0^{1,\infty}(Q; \mathbb{R}^n).$

- $F: \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ **symmetric rank-one-convex**

$\iff \mathbb{R} \ni t \mapsto F(A + ta \odot b) \text{ convex for all } A \in \mathbb{R}_{\text{sym}}^{n \times n}, a, b \in \mathbb{R}^n.$

- Recession function: $F^\infty(z) := \lim_{t \rightarrow \infty} \frac{1}{t} F(tz)$

Have: Convexity \implies Symmetric Quasiconvexity \implies Symmetric Rank-1-Convexity

Theorem (Rindler, '11)

If $F \in C(\mathbb{R}_{\text{sym}}^{n \times n})$ is SQC & of linear growth and $u_0 \in \text{BD}(\Omega)$, then the functional $u \mapsto \overline{\mathcal{F}}[u]$ is sequentially weak*-lower semicontinuous on BD .

→ Existence OK – Regularity?

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Partial Regularity for BV-Minimisers

- For existence of minima, QC of F does not suffice \implies Strong QC of F :

$$\mathbb{R}^{N \times n} \ni \xi \mapsto F(\xi) - \ell E(\xi) \text{ is QC, where } E(\xi) := \sqrt{1 + |\xi|^2} - 1.$$

Theorem (FXG & Kristensen, 2018)

Let $F \in C_{\text{loc}}^{2,1}(\mathbb{R}^{N \times n})$ be a variational integrand that is

- ① strongly quasiconvex
- ② of linear growth: $|F(z)| \lesssim 1 + |z|$ for all $z \in \mathbb{R}^{N \times n}$.

Then any generalised local minimiser $u \in BV(\Omega; \mathbb{R}^N)$ for the variational integral

$$\mathcal{F}[v] := \int_{\Omega} F(Dv), \quad v \in BV(\Omega; \mathbb{R}^N)$$

is $C^{1,\alpha}$ -partially regular.

- precursor by ANZELLOTTI & GIAQUINTA '88 in the **convex** case.
- extension of the theorem possible in several directions, above is the simplest version.

Caccioppoli of the 2nd kind

- As usual, we aim for an **excess decay**. First step:

Caccioppoli Inequality

For every $m > 0$ there exists $c = c(m) \in [1, \infty)$ such that if u is a generalised minimiser, then for all $a: \mathbb{R}^n \rightarrow \mathbb{R}^N$ affine with $|Da| \leq m$ and $B(x, 2R) \Subset \Omega$:

$$\int_{B(x, R)} E(Du) \leq c \int_{B(x, 2R)} E\left(\frac{u - a}{R}\right) dx$$

- Proof standard
- Excludes oscillation **but only limits concentration**
- Does **not** imply higher integrability of Du (!)

Linearisation

- Linearisation: Locally the integrand behaves almost quadratically
→ harmonic approximation only on good balls!
- Shifted Integrands:

$$F_w(z) := F(w + z) - F(w) - \langle F'(w), z \rangle, \quad w, z \in \mathbb{R}^{N \times n}.$$

- Excess decay where $|(Du)_{B(x,R)}| \leq M$ is controlled large:
 - $|(Du)_{B(x,R)}| \leq M < \infty$.
 - $u|_{\partial B(x,R)} \in W^{\frac{1}{n}, \frac{n}{n-1}}(B(x, R); \mathbb{R}^N)$
 - $Du|_{\partial B(x,R)} \equiv 0$

Linearisation Continued

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- Let $a: \mathbb{R}^n \rightarrow \mathbb{R}^N$ be affine with $\nabla a = (Du)_{B(x,R)}$, $\tilde{u} := u - a$ and $\tilde{F} = F_{\nabla a}$. Then

$$\int_{B(x,R)} \langle \tilde{F}'(\nabla \tilde{u}), \nabla \varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in W_0^{1,1}(B(x, R); \mathbb{R}^N).$$

Linearisation Continued

- Then for all $\varphi \in C_0^1(B(x, R); \mathbb{R}^N)$:

$$\begin{aligned}
\int_{B(x, R)} \langle \tilde{F}''(0)D\tilde{u}, \nabla \varphi \rangle &= \int_{B(x, R)} \langle \tilde{F}''(0)\nabla \tilde{u}, \nabla \varphi \rangle dx \\
&\quad + c|D^s u|(B(x, R)) \sup_{B(x, R)} |\nabla \varphi| \\
&= \int_{B(x, R)} \langle \tilde{F}''(0)\nabla \tilde{u} - \tilde{F}'(\nabla \tilde{u}), \nabla \varphi \rangle dx \\
&\quad + c|D^s u|(B(x, R)) \sup_{B(x, R)} |\nabla \varphi| \\
&\leq c(\sup_{B(x, R)} |\nabla \varphi|) \int_{B(x, R)} E(D\tilde{u}).
\end{aligned}$$

- $\mathbb{A} := \tilde{F}''(0) = F''(\nabla a)$ is Legendre-Hadamard elliptic.
- If $u|_{\partial B(x, R)} \in W^{\frac{1}{n}, \frac{n}{n-1}}(\partial B(x, R); \mathbb{R}^N)$, then

$$\begin{cases} -\operatorname{div}(A\nabla h) = 0 & \text{in } B(x, R), \\ h = u & \text{on } \partial B(x, R) \end{cases}$$

has a unique solution $h \in W^{1, \frac{n}{n-1}}(B(x, R); \mathbb{R}^N)$.

\implies For all $\varphi \in (C^1 \cap W_0^{1,\infty})(B(x, R); \mathbb{R}^N)$:

$$\int_{B(x, R)} \mathbb{A}[D(\tilde{u} - h), \nabla \varphi] \leq c \left(\sup_{B(x, R)} |\nabla \varphi| \right) \int_{B(x, R)} E(D\tilde{u})$$

- Put $\psi := \tilde{u} - h \in \text{BV}_0(\text{B}(x, R); \mathbb{R}^N)$.
- Truncation map:

$$T(y) := \begin{cases} y & \text{if } |y| \leq 1, \\ y/|y| & \text{otherwise} \end{cases} \implies T(\psi) \in L^\infty \cap \text{BV}_0(\text{B}(x, R); \mathbb{R}^N)$$

\implies If $\varphi \in W_0^{1,2}(\text{B}(x, R); \mathbb{R}^N)$ solves

$$-\operatorname{div}(\mathbb{A}\nabla\varphi) = T(\psi) \implies \forall p < \infty: \varphi \in W^{2,p}(\text{B}(x, R); \mathbb{R}^N),$$

$$\|\nabla\varphi\|_{L^\infty} \leq C\|T(\psi)\|_{L^p} \leq C \left(\int_{\text{B}(x, R)} E(\psi) \right)^{\frac{1}{p}}.$$

$$\begin{aligned} \Rightarrow \int_{\text{B}} E(D\psi) &\leq \int_{\text{B}} \langle T(\psi), \psi \rangle = \int_{\text{B}} \langle \mathbb{A}\nabla\varphi, D\psi \rangle \\ &= \int_{\text{B}} \langle \mathbb{A}D\psi, \nabla\varphi \rangle \leq C \left(\int_{\text{B}(x, R)} E(\psi) \right)^{\frac{1}{p}} \int_{\text{B}(x, R)} E(D\tilde{u}) \end{aligned}$$

Conclusion

- yields

$$\fint_{B(x_0, R)} E\left(\frac{\tilde{u} - h}{R}\right) dx \leq C \left(\fint_{B(x_0, R)} E(D\tilde{u}) \right)^q, \quad 1 < q < \frac{n}{n-1}$$

and in combination with Caccioppoli, we obtain

$$E(x_0, \sigma R_0) \lesssim \left(\frac{1}{\sigma^2} \left(\frac{E(x_0, R_0)}{R_0^n} \right)^{q-1} + \sigma^{n+2} \right) E(x_0, R_0).$$

- Now semi-standard iteration completes the proof.

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The Symmetric Quasiconvex Case

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- extension of the theorem possible in several directions, above is the simplest version.

Comparison 1: Selection of *good* radii

- Comparison of $u \in \text{GM}(F)$ with suitably \mathbb{A} -harmonic maps on **good** balls.
 - Regularity of comparison maps is dictated by the interior traces of u
 - + need a boundary regularity for **symmetric Legendre-Hadamard systems**.
- again, crucial estimate:

$$\fint_{B(x_0, R)} V\left(\frac{\tilde{u} - h}{R}\right) dx \leq C \left(\fint_{B(x_0, R)} V(E\tilde{u}) \right)^q, \quad 1 < q < \frac{n+1}{n}$$

Reduction Principle: If $\mathbb{A}: \mathbb{R}_{\text{sym}}^{n \times n} \times \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ is a SSR1-convex BLF, then ...

$\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \ni (\xi, \eta) \mapsto \mathcal{A}[\xi, \eta] := \mathbb{A}[\xi^{\text{sym}}, \eta^{\text{sym}}]$ is in fact SSR1-convex!

→ Maz'ya & Shaposhnikova step in and give that

$$\Phi: W^{k,q}(B; \mathbb{R}^n) \ni u \mapsto (-\text{div}(\mathbb{A} Eu), \text{Tr}_{\partial B}(u))$$

$$\in W^{k-2,q}(B; \mathbb{R}^n) \times W^{k-\frac{1}{q},q}(\partial B; \mathbb{R}^n)$$

is a toplinear isomorphism.

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→ Maz'ya & Shaposhnikova step in and give for $q = \frac{n+1}{n}$ and $k = 1$

$$\Phi: W^{1, \frac{n+1}{n}}(B; \mathbb{R}^n) \ni u \mapsto (0, \text{Tr}_{\partial B}(u))$$

$$\in W^{k-2, q}(B; \mathbb{R}^n) \times W^{\frac{1}{n+1}, \frac{n+1}{n}}(\partial B; \mathbb{R}^n)$$

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Intermezzo: Boundary Values & Embeddings for BD-maps

- If $\Omega \subset \mathbb{R}^n$ is open, bounded with $n \geq 2$ & Lipschitz, then $\text{BV}(\Omega)$ has trace space $L^1(\partial\Omega)$ & $\text{BV}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$
- Strauss '73 $\implies \text{BD}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$
- The Trace Embedding for BD is due to Babadjan '13 (based on Strang-Temam '80) (!)
- Kolyada '96 + Bourgain, Brezis & Mironescu '02:

$$n \geq 2 \Rightarrow \text{BV}(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}(\Omega), \quad \theta \in (0, 1)$$
- Van Schaftingen '13, FXG, Kristensen '18:

$$\Rightarrow n \geq 2 \Rightarrow \text{BD}(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}(\Omega), \quad \theta \in (0, 1)$$

Moral of the story

Ornstein is (almost) invisible on the level of fractional derivatives!

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Intermezzo: Boundary Values & Embeddings for BD-maps

- If $\Omega \subset \mathbb{R}^n$ is open, bounded with $n \geq 2$ & Lipschitz, then $\text{BV}(\Omega)$ has trace space $L^1(\partial\Omega)$ & $\text{BV}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$
- Strauss '73 $\implies \text{BD}(\Omega) \hookrightarrow L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n)$
- The Trace Embedding for BD is due to Babadjanian '13 (based on Strang-Temam '80) (!)
- Kolyada '96 + Bourgain, Brezis & Mironeanu '02:

$$n \geq 2 \Rightarrow \text{BV}(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}(\Omega), \quad \theta \in (0, 1)$$
- Van Schaftingen '13, FXG, Kristensen '18:

$$\Rightarrow n \geq 2 \Rightarrow \text{BD}(\Omega) \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}(\Omega), \quad \theta \in (0, 1)$$

Moral of the story

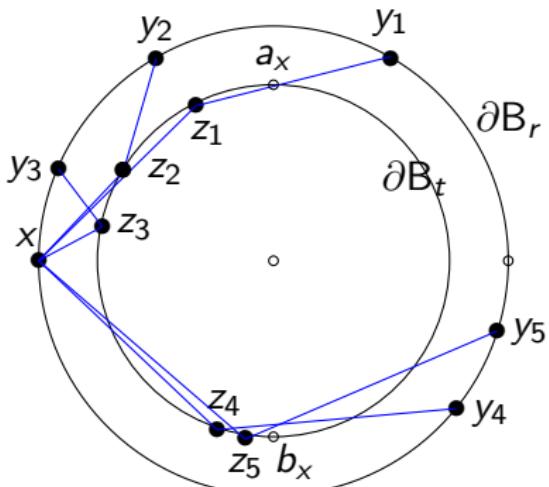
Ornstein is (almost) invisible on the level of fractional derivatives!

Comparison 2: Fubini-type theorem for BD-maps

$$\int_0^R \iint_{\partial B(x_0, r) \times \partial B(x_0, r)} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp-1}} d\sigma_y d\sigma_x dr \leq C \iint_{B(x_0, R) \times B(x_0, R)} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy$$

Comparison 2: Fubini-type theorem for BD-maps

$$\longrightarrow \text{estimate } \int_0^R (r^{n-1})^2 \iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|u(rx) - u(ry)|^p}{|rx - ry|^{n+sp-1}} d\sigma_y d\sigma_x dr$$



- projections $\pi_t(x, y)$:

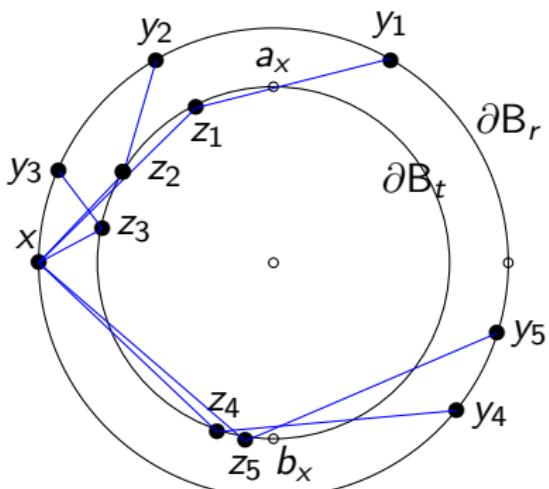
$$|u(rx) - u(ry)|^p \lesssim (|u(rx) - u(\pi_t(x, y))|^p + |u(ry) - u(\pi_t(x, y))|^p)$$

- integration with respect to t :

$$|u(rx) - u(ry)|^p \lesssim \int_{a(x,y)}^{b(x,y)} |u(rx) - u(\pi_t(x, y))|^p dt + \int_{a(x,y)}^{b(x,y)} |u(ry) - u(\pi_t(x, y))|^p dt.$$

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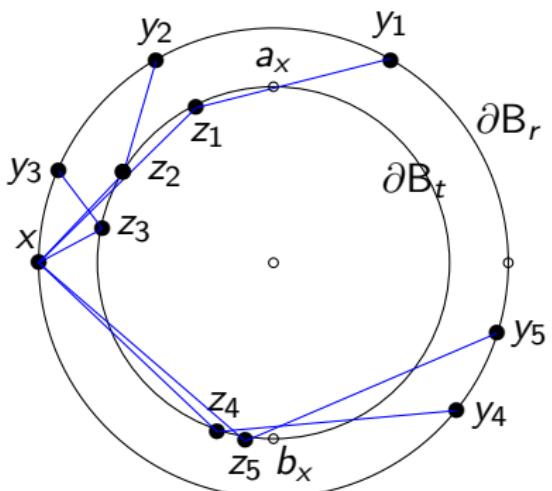
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The Implications

- Embed $\text{BD} \hookrightarrow W^{\theta, \frac{n}{n-1+\theta}}$ (Van Schaftingen '13, Kristensen & FXG '17)
- Sufficiently many Lebesgue points / Lebesgue spheres – uses Kohn '81:

$$\Sigma \subset \mathbb{R}^n \quad C^1\text{-hypersurface} \Rightarrow Eu \llcorner \Sigma = (u^+ - u^-) \odot \nu_\Sigma \mathcal{H}^{n-1} \llcorner \Sigma$$

$\implies W^{s,p}(\partial B(x_0, r))$ -norm of u can be controlled via $|Eu|(B(x_0, R))$, $r < R$ suitably close.

- finally yields (+Caccioppoli of 2nd kind)

$$E(x_0, \sigma R_0) \lesssim \left(\frac{1}{\sigma^2} \left(\frac{E(x_0, R_0)}{R_0^n} \right)^{\frac{q-1}{q}} + \sigma^{n+2} \right) E(x_0, R_0).$$

\implies now semi-standard iteration concludes the proof.

Thank you!