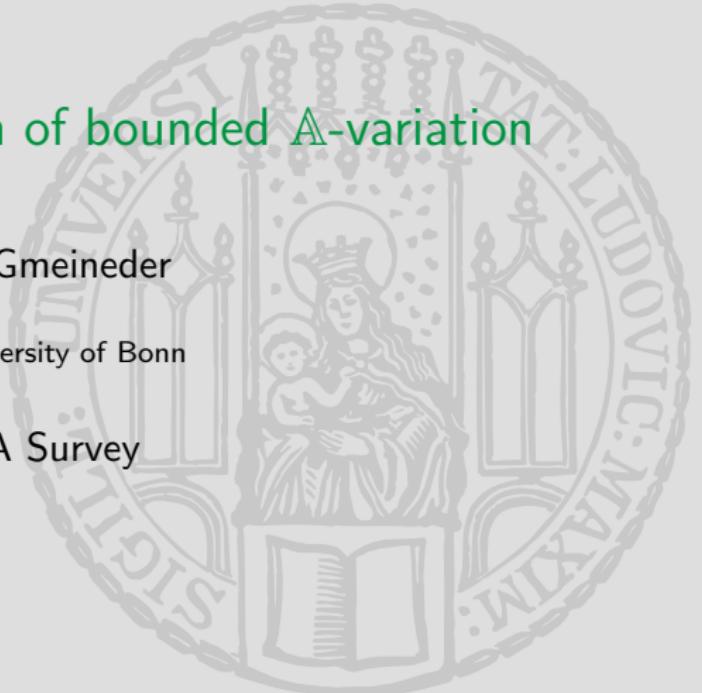


Traces for function of bounded \mathbb{A} -variation

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A Survey



Intro & Setup

- Let V, W finite dimensional \mathbb{R} -vector spaces, $\mathbb{A}_k \in \mathcal{L}(V; W)$ and put

$$\mathbb{A}[D] = \sum_{k=1}^n \mathbb{A}_k \partial_k.$$

- We call $\mathbb{A}[D]$ **elliptic** if and only if

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}: \mathbb{A}[\xi] := \sum_{k=1}^n \mathbb{A}_k \xi_k, \quad \xi = (\xi_1, \dots, \xi_n),$$

is an injective map $\mathbb{A}[\xi]: V \rightarrow W$. We put, for $\Omega \subset \mathbb{R}^n$ open,

$$W^{\mathbb{A},p}(\Omega) := \{u \in L^p(\Omega; V): \mathbb{A}[D]u \in L^p(\Omega; W)\}.$$

→ When $W^{\mathbb{A},p} \simeq W^{1,p}$? If not, which properties survive?

Ornstein versus Korn I

Let $\varphi \in \mathcal{S}(\mathbb{R}^n; V)$ and put $f := \mathbb{A}[D]\varphi$. Then

$$\begin{aligned}\mathbb{A}[\xi]\widehat{\varphi}(\xi) &= \widehat{f}(\xi) \Rightarrow \mathbb{A}^*[\xi]\mathbb{A}[\xi]\widehat{\varphi}(\xi) = \mathbb{A}^*[\xi]\widehat{f}(\xi) \\ &\Rightarrow \widehat{\varphi}(\xi) = (\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1}\mathbb{A}[\xi]\widehat{f}(\xi) \\ &\Rightarrow \varphi(x) = \mathcal{F}^{-1}[(\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1}\mathbb{A}[\xi]\widehat{\mathbb{A}[D]\varphi}(\xi)] \\ &\Leftrightarrow \varphi = \mathcal{F}^{-1}[\color{red}m(\xi)\color{black}\widehat{\mathbb{A}[D]\varphi}(\xi)] =: \Phi_{\mathbb{A}}(\mathbb{A}[D]\varphi).\end{aligned}$$

- multiplier m is homogeneous of degree (-1) .
- yields

$$\Phi_{\mathbb{A}}(\mathbb{A}[D]\varphi) \sim \int_{\mathbb{R}^n} \frac{\mathbb{A}[D]\varphi(y)}{|\cdot - y|^{n-1}} dy \quad \& \quad \Rightarrow \Phi_{\mathbb{A}}: L^1 \rightarrow L^{\frac{n}{n-1}, \infty} \supsetneq L^{\frac{n}{n-1}}$$

Ornstein versus Korn II

Let $\varphi \in \mathcal{S}(\mathbb{R}^n; V)$ and put $f := \mathbb{A}[D]\varphi$. Then

$$\begin{aligned}\mathbb{A}[\xi]\widehat{\varphi}(\xi) &= \widehat{f}(\xi) \Rightarrow \mathbb{A}^*[\xi]\mathbb{A}[\xi]\widehat{\varphi}(\xi) = \mathbb{A}^*[\xi]\widehat{f}(\xi) \\ \stackrel{\forall j}{\Rightarrow} \xi_j \widehat{\varphi}(\xi) &= (\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1}\mathbb{A}[\xi]\widehat{f}(\xi) \\ \Rightarrow \partial_j \varphi(x) &= \mathcal{F}^{-1}[\xi_j(\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1}\mathbb{A}[\xi]\widehat{\mathbb{A}[D]\varphi}(\xi)] \\ \Leftrightarrow \partial_j \varphi &= \partial_j \varphi(x) = \mathcal{F}^{-1}[\tilde{m}_j(\xi)\widehat{\mathbb{A}[D]\varphi}(\xi)] = D\Phi_{\mathbb{A}}(\mathbb{A}[D]\varphi).\end{aligned}$$

- multiplier \tilde{m}_j is homogeneous of degree zero.
- yields

$$D\Phi_{\mathbb{A}}(\mathbb{A}[D]\varphi) \sim p.v. \int_{\mathbb{R}^n} \frac{\mathbb{A}[D]\varphi(y)}{|\cdot - y|^n} dy \quad \& \quad \Rightarrow \Phi_{\mathbb{A}}: L^1 \not\rightarrow L^1$$

Ornstein versus Korn III

Theorem (Ornstein '62, Kirchheim & Kristensen '16)

Let V, W_1, W_2 be finite-dimensional inner product spaces, $k \in \mathbb{N}$ and let

$$\mathbb{A}_i[D] = \sum_{|\alpha|=k} a_\alpha^i \partial^\alpha, \quad i = 1, 2,$$

where $a_\alpha^i \in \mathcal{L}(V; W_i)$. Then the following are equivalent:

- There exists $c > 0$ such that

$$\|\mathbb{A}_1[D]\varphi\|_{L^1} \leq c \|\mathbb{A}_2[D]\varphi\|_{L^1} \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n; V).$$

- There exists $C \in \mathcal{L}(W_1; W_2)$ such that

$$a_\alpha^2 = C a_\alpha^1 \quad \text{for all } \alpha \in \mathbb{N}_0^n, |\alpha| = k.$$

Bourgain-Brezis-Estimates

Consider

$$-\Delta u = f \in L^1(\mathbb{R}^n; \mathbb{R}^n)$$

- In general, we do not have $Du \in L^{n/(n-1)}(\mathbb{R}^n)$:

$$u(x) \sim \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy \implies Du(x) \sim \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-1}} dy =: \mathcal{I}_1(f),$$

but $\mathcal{I}_1: L^1(\mathbb{R}^n; \mathbb{R}^n) \rightarrow L^{\frac{n}{n-1}, \infty}(\mathbb{R}^n; \mathbb{R}^n) \not\subset L^{\frac{n}{n-1}}(\mathbb{R}^n; \mathbb{R}^n)$!

- Bourgain & Brezis: **But sometimes we do!**

→ If $\operatorname{div}(f) = 0$, then in fact $Du \in L^{\frac{n}{n-1}}(\mathbb{R}^n; \mathbb{R}^n)$.



Recasting the questions

- What is a necessary and sufficient condition on \mathbb{A} for the embedding

$$W^{\mathbb{A},1}(\mathbb{R}^n) \hookrightarrow L^{\frac{n}{n-1}}(\mathbb{R}^n; V)$$

to hold true?

- Let $\Omega \subsetneq \mathbb{R}^n$. What is a necessary and sufficient condition on \mathbb{A} for the embedding

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- Let $\Omega \subsetneq \mathbb{R}^n$. What is a necessary and sufficient condition on \mathbb{A} for the boundary trace embedding

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The case 1 < p < ∞

- Consider the *trace-free symmetric gradient*

$$\nabla^{\text{tfsym}} u := \nabla^{\text{sym}} u - \frac{1}{n} \operatorname{div}(u) E_n.$$

- In $n = 2$ dimensions, we have

$$\nabla^{\text{tfsym}} u = \frac{1}{2} \begin{pmatrix} \partial_1 u_1 - \partial_2 u_2 & \partial_1 u_2 + \partial_2 u_1 \\ \partial_1 u_2 + \partial_2 u_1 & \partial_2 u_2 - \partial_1 u_1 \end{pmatrix}$$

- $\mathbb{R}^2 \simeq \mathbb{C} \Rightarrow$ holomorphic functions $\mathbb{D} \rightarrow \mathbb{C}$ contained in $\ker(\nabla^{\text{tfsym}})$!
- Take $f(z) := \frac{1}{z-1}$, then

$$\int_{\partial\mathbb{D}} |f| d\mathcal{H}^1 = +\infty.$$

Boundary traces for $1 < p < \infty$

Theorem (Smith, '69 + Raita & G, '17)

Let $1 < p < \infty$ and denote B the open unit ball in \mathbb{R}^n . The following are equivalent:

1. $W^{\mathbb{A},p}(B) \simeq W^{1,p}(B; V)$.
2. There exists a bounded linear extension operator

$$E: W^{\mathbb{A},p}(B) \rightarrow W^{\mathbb{A},p}(\mathbb{R}^n).$$

3. \mathbb{A} has finite dimensional nullspace.
4. \mathbb{A} is \mathbb{C} -elliptic.
5. $W^{\mathbb{A},p}(B) \hookrightarrow L^q(B; V)$ for some $q > p$.
6. $\text{tr}(W^{\mathbb{A},p}(B); \partial B) = W^{1-\frac{1}{p},p}(\partial B; V)$.



Trace Embeddings

Theorem (Adams, Maz'ya)

Let $1 < p < q < \infty$. A necessary and sufficient condition for the map $f \mapsto \mathcal{I}_\alpha(f)$ to be continuous from $L^p(\mathbb{R}^n) \rightarrow L^q(\mu; \mathbb{R}^n)$ is that

$$\sup_{x,r>0} \frac{\mu(B(x,r))}{\dot{C}_{\alpha,p}(B(x,r))^{q/p}} < \infty.$$

- Suppose that for $0 < s < p$ with $sp < n$ we put $\mu = \mathcal{H}^{n-s} \llcorner \Sigma$.

$$\Rightarrow \dot{C}_{\alpha,p}(B(x,r)) \sim r^{n-\alpha p} \Rightarrow \dot{C}_{\alpha,p}(B(x,r))^{q/p} \sim r^{q(n-\alpha p)/p}.$$

$$\Rightarrow \sup_{x,r>0} r^{n-s-q(n-\alpha p)/p} < \infty \Leftrightarrow n - s - q(n - \alpha p)/p = 0 \Leftrightarrow q = p \frac{n - s}{n - \alpha p}.$$



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Corollary

Let $\mathbb{A}[D]$ be elliptic. Consider a decomposition of the form

$$u = \Pi_{\mathbb{A}}(u) + T\mathbb{A}[D]u,$$

where

- $\Pi_{\mathbb{A}} : W^{\mathbb{A},p}(B) \rightarrow \ker(\mathbb{A})$ is a projection,
 - T is a Riesz potential operator of order $(n - 1)$.
- \Rightarrow then $u - \Pi_{\mathbb{A}}(u)$ attains 'good traces'!
- Conjecture: It is only the boundary behaviour of \mathbb{A} -free L^p -integrable maps which leads to the failure of usual embeddings – also for $p = 1$.

EC – Ellipticity & Cancellation

Definition (Cancelling Operators)

A differential operator $\mathbb{A}[D] = \sum_{k=1}^n \mathbb{A}_k \partial_k$ with $\mathbb{A}_k: V \rightarrow W$ is called **cancelling** provided

$$\bigcap_{\xi \in \mathbb{R}^n \setminus \{0\}} \mathbb{A}[\xi](V) = \{0\}.$$

If $\mathbb{A}[D]$ is **both elliptic and cancelling**, then we say that $\mathbb{A}[D]$ is **EC**.

Theorem (Van Schaftingen, JEMS '13)

The following are equivalent for a differential operator $\mathbb{A}[D] = \sum_{k=1}^n \mathbb{A}_k \partial_k$:

1. $\exists c > 0 \forall u \in C_c^\infty(\mathbb{R}^n; V): \|u\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n; V)} \leq c \|\mathbb{A}[D]u\|_{L^1(\mathbb{R}^n; W)}$.
2. $\mathbb{A}[D]$ is EC.

Van Schaftingen's Proof

- Elliptic estimates: If \mathbb{A} is elliptic, then

$$\|u\|_{L^{\frac{n}{n-1}}} \leq c \|\mathbb{A}[D]u\|_{\dot{W}^{-1,n}}.$$

- Key point: If \mathbb{A} is elliptic & cancelling, then

$$\left| \int_{\mathbb{R}^n} \varphi \mathbb{A}[D]u \, dx \right| \leq c \|\mathbb{A}[D]u\|_{L^1} \|D\varphi\|_{L^n}$$

- The previous estimate just asserts

$$\|\mathbb{A}[D]u\|_{\dot{W}^{-1,n}} \leq C \|\mathbb{A}[D]u\|_{L^1},$$

and this completes the proof.

Boundary Traces for $p = 1$ - Gradient & Symmetric Gradient

- GAGLIARDO, 60's + ANZELLOTTI & GIAQUINTA, '79:

$$\text{tr}(\mathcal{W}^{1,1}(\Omega); \partial\Omega) = \text{tr}(\mathcal{BV}(\Omega); \partial\Omega) = \mathcal{L}^1(\partial\Omega).$$

- STRANG & TEMAM '81, BABADJIAN '13:

$$\text{tr}(\mathcal{LD}(\Omega); \partial\Omega) = \text{tr}(\mathcal{BD}(\Omega); \partial\Omega) = \mathcal{L}^1(\partial\Omega; \mathbb{R}^n).$$

Here,

$$\mathcal{LD}(\Omega) := \{ u \in \mathcal{L}^1(\Omega; \mathbb{R}^n) : \nabla^{\text{sym}} u \in \mathcal{L}^1(\Omega; \mathbb{R}^{n \times n}) \},$$

$$\mathcal{BD}(\Omega) := \{ u \in \mathcal{L}^1(\Omega; \mathbb{R}^n) : \nabla^{\text{sym}} u \in \mathcal{M}_{<\infty}(\Omega; \mathbb{R}^{n \times n}) \}.$$

Boundary traces for $p = 1$

Definition

Let $\Omega \subset \mathbb{R}^n$ be open. We define

$$\text{BV}^{\mathbb{A}}(\Omega) := \{u \in L^1(\Omega; V) : \mathbb{A}[D]u \in \mathcal{M}_{<\infty}(\Omega; W)\}.$$

Theorem (Breit, Diening, G '17)

Let $\Omega \subset \mathbb{R}^n$ be open, bounded with Lipschitz boundary. The following are equivalent:

- $\text{tr}(\text{BV}^{\mathbb{A}}(\Omega); \partial\Omega) = \text{tr}(W^{\mathbb{A},1}(\Omega); \partial\Omega) = L^1(\partial\Omega; V).$
- $\mathbb{A}[D]$ is \mathbb{C} -elliptic.

Why important at all? - 'Practical Need'

- Variational Problems: minimise

$$\mathcal{F}[u] := \int_{\Omega} f(\mathbb{A}[D]u) \, dx \quad \text{over a } \underline{\text{Dirichlet class}}$$

- f is of **linear growth**, i.e.,

$$|\cdot| - 1 \lesssim f(\cdot) \lesssim |\cdot| + 1.$$

- Even for BV \rightarrow trace operator *not weak*-continuous!*

\rightarrow pass to relaxation

$$\begin{aligned} \overline{\mathcal{F}}[u] = & \int_{\Omega} f(\nabla u) \, dx + \int_{\Omega} f^{\infty} \left(\frac{dD^s u}{d|D^s u|} \right) d|D^s u| \\ & + \int_{\partial\Omega} f^{\infty} (\operatorname{tr}(u_0 - u) \otimes \nu_{\partial\Omega}) \, d\mathcal{H}^{n-1} \end{aligned}$$

Why important at all? - 'Practical Need'

- Needed: If $u \in BV^{\mathbb{A}}(\Omega)$, then $\bar{u} \in BV^{\mathbb{A}}(\mathbb{R}^n)$, where

$$\bar{u} := \begin{cases} u & \text{in } \Omega, \\ u_0 & \text{in } \mathbb{R}^n \setminus \bar{\Omega}. \end{cases}$$

- Problem:** Is $\mathbb{A}\bar{u} \in \mathcal{M}_{<\infty}(\mathbb{R}^n; W)$?
- DE PHILIPPIS & RINDLER, '16, ANNALS: If

$$\mathcal{A} = \sum_{|k|=\ell} \mathcal{A}_k \partial_k$$

is a constant rank differential operator with $\mathcal{A}_k \in \mathcal{L}(W; X)$, then

$$(\mu \in \mathcal{M}(\mathbb{R}^n; W) \text{ & } \mathcal{A}\mu \equiv 0) \Rightarrow \frac{d\mu}{d|\mu^s|} \in \Lambda_{\mathcal{A}} = \bigcup_{\xi \neq 0} \ker(\mathcal{A}[\xi]) \text{ } |\mu^s|\text{-a.e.}$$

→ this is a notable generalisation of ALBERTI's rank-one-theorem.

Setting

- Elliptic complex:

$$V \xrightarrow{\mathbb{A}[\xi]} W \xrightarrow{\mathcal{A}[\xi]} X$$

exact at W for all $\xi \neq 0$.

- Examples:

1. $\mathbb{A}[D] = \nabla$, $\mathcal{A}[D] = \text{curl}$.
2. $\mathbb{A}[D] = \nabla^{\text{sym}}$, $\mathcal{A}[D] = \text{curl curl}$. Here,

$$\text{curl curl}(E) = \left(\sum_{i=1}^n \partial_{ik} E_i^j + \partial_{ij} E_i^k - \partial_{jk} E_i^i - \partial_{ii} E_j^k \right)_{j,k=1,\dots,n}.$$

- De Philippis-Rindler Theorem only applicable if we know that

$$\mathbb{A}\bar{u} \in \mathcal{M}(\mathbb{R}^n; W).$$

Sketch of the Proof (Half Space)

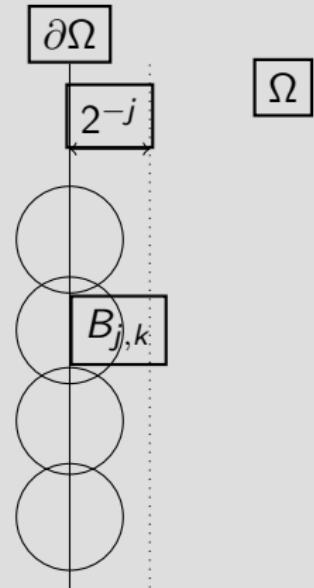
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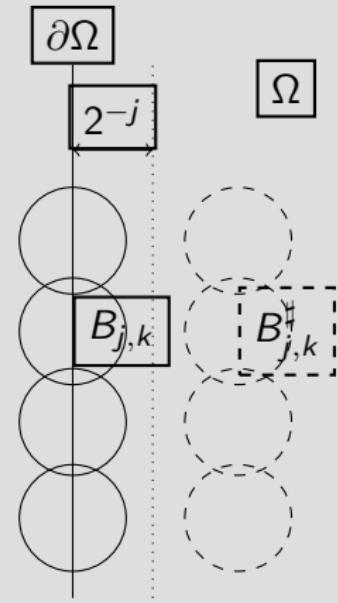


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For every $B_{j,k}$ choose reflected ball $B_{j,k}^\sharp$ in Ω .



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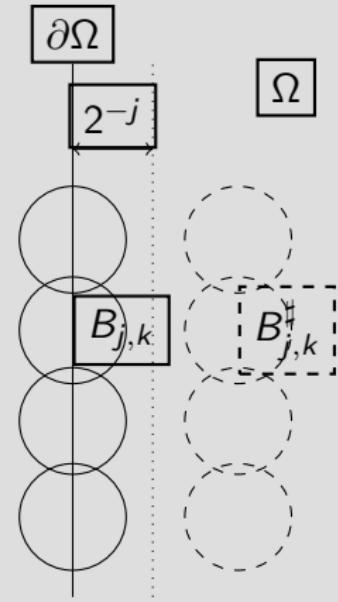
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Project u on $B_{i,k}^\#$ to $\Pi_{j,k} u \in N(\mathbb{A})$ and

replace u on $B_{j,k}$ by $\Pi_{j,k} u$ i.e.

$$T_j u := (1 - \rho_j)u + \rho_j \sum_k \eta_{j,k} \Pi_{j,k} u.$$



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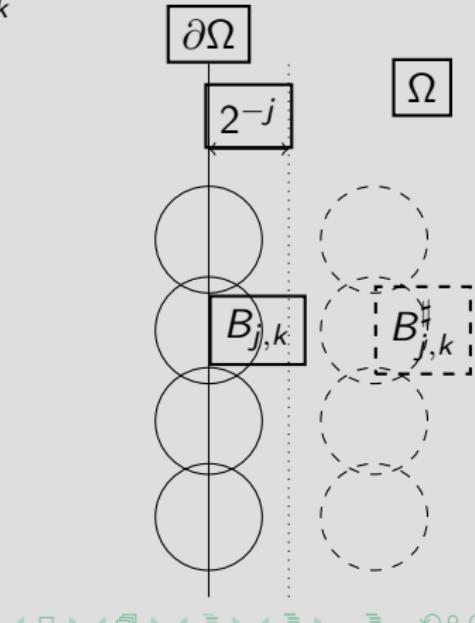
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Then $T_j u \rightarrow u$ in $BV^{\mathbb{A}}(\Omega)$

and $\text{tr}(T_j u) \rightarrow: \text{tr}(u)$ in $L^1(\partial\Omega)$.

Based on inverse estimates for polynomials!



Embeddings on domains

The following are equivalent:

- $W^{\mathbb{A},1}(B) \hookrightarrow L^{\frac{n}{n-1}}(B; V)$.
- \mathbb{A} is \mathbb{C} -elliptic.

Would expect: If \mathbb{A} is elliptic and cancelling, then there exists a trace operator

$$\text{tr}: (W^{\mathbb{A},1} / \ker(\mathbb{A}))(\Omega) \rightarrow L^1(\partial\Omega; V)$$



Thanks a lot for the attention!