On the Characterisation of Korn–type Operators

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Introduction & Overview

Consider the variational integral

$$\mathfrak{F}[v] := \int_{\Omega} f(\varepsilon(u))\,\mathrm{d}x, \qquad u\colon \Omega \to \mathbb{R}^n,$$

where

- Ω is an open and bounded Lipschitz subset of $\mathbb{R}^n$,
- $\varepsilon(u) := \frac{1}{2}(Du + D^{\mathsf{T}}u)$ is the **symmetric gradient**, and
- $f\colon \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}_{\geq 0}$ is continuous and satisfies for $1 < p < \infty$

$$c_1|\xi|^p \leq f(\xi) \leq c_2(1+|\xi|)^p \qquad \text{for all } \xi \in \mathbb{R}^{n\times n}_{\mathrm{sym}}.$$

- Given $u_0 \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^n)$, can we prove existence of minima of $\mathfrak{F}$ in the Dirichlet class $\mathcal{D}_{u_0} := u_0 + \mathrm{W}^{1,p}_0(\Omega;\mathbb{R}^n)$?

Korn–type Inequalities: $1 < p < \infty$

- Crucial Ingredient: Korn–type Inequalities allow to estimate the **full** gradient against the **symmetric part** of the gradient, e.g.,

$$\exists C > 0\, \forall \varphi \in \mathrm{C}_c^\infty(\Omega;\mathbb{R}^n)\colon\ \|D\varphi\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})} \leq C\|\varepsilon(\varphi)\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})}.$$

- Important: Restriction to $1 < p < \infty$ is necessary!
- Other Korn–type Inequalities also available:

$$\exists C > 0\, \forall \varphi \in \mathrm{C}^\infty(\Omega;\mathbb{R}^n)\, \exists \Pi \in \mathcal{R}(\Omega)\colon \\ \|D(\varphi - \Pi)\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})} \leq C\|\varepsilon(\varphi)\|_{\mathrm{L}^p(\Omega;\mathbb{R}^{n\times n})},$$

where $\mathcal{R}(\Omega) := \{x \mapsto Ax + b\colon\ A \in \mathbb{R}^{n\times n}_{\mathrm{scew}},\, b \in \mathbb{R}^n\}$ is the nullspace of ε.

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Need for a unifying approach

- In a similar vein: If we replace ε by other differential operators, what survives?
- E.g., in compressible fluid mechanics, one often considers the deviatoric part of the symmetric gradient

$$\varepsilon^D(u) := \varepsilon(u) - \frac{1}{n}\operatorname{div}(u)\mathrm{Id}, \qquad n \geq 3$$

$\longrightarrow$ **Goal:** A unifying approach to Korn–type Inequalities which does not depend on the specific differential operator.

- Given two finite–dimensional $\mathbb{R}$–vector spaces V, W, a **standard operator** is a first order, linear, homogeneous and constant coefficient differential operator of the form

$$\mathbb{A}[D] := \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=1}} \mathbb{A}_\alpha \partial^\alpha \text{ with } \mathbb{A}_\alpha \in \mathscr{L}(V, W) \text{ fixed.}$$

Old and New Questions

- Consider the variational integral

$$\mathfrak{F}[v] := \int_\Omega f(\mathbb{A}[D]v)\,\mathrm{d}x.$$

with an integrand $f \in \mathrm{C}^2(W;\mathbb{R})$ of p–growth, $p > 1$. **Questions:**
  - What is the correct function space framework?
  - When can we reduce this framework to the usual Sobolev spaces?
- As we will find out, the correct notion is that of **FDP**–operators, i.e., differential operators $\mathbb{A}[D]$ with

$$\dim(\ker(\mathbb{A}[D];\mathrm{B})) < \infty.$$

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Korn, Ornstein and all that, on $\mathbb{R}^n$ – I

Suppose we want to prove

$$\forall 1 < p < \infty \, \exists C > 0 \, \forall \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)\colon \; \|D\varphi\|_{\mathrm{L}^p} \leq C\|\mathbb{A}[D]\varphi\|_{\mathrm{L}^p}$$

and suppose there is a linear map

$$\varPhi\colon \; \mathrm{C}_c^\infty(\mathbb{R}^n; W) \ni \mathbb{A}[D]\varphi \mapsto D\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n; \mathbb{R}^n \times V).$$

$\longrightarrow$ What are the mapping properties of $\varPhi$? Does $\varPhi$ boundedly extend to a continuous linear map on $\mathrm{L}^p(\mathbb{R}^n; W)$?

$\longrightarrow$ As we will see, $\varPhi$ is a **singular integral of convolution type** and thus is bounded between $\mathrm{L}^p \to \mathrm{L}^p$ **if and only if** $1 < p < \infty$.

Korn, Ornstein and all that, on $\mathbb{R}^n$ – II

- A differential operator of the form $\mathbb{A}[D] = \sum_{k=1}^{n} \mathbb{A}_k \partial_k$ with $\mathbb{A}_k \in \mathscr{L}(V;E)$ fixed is called **elliptic** provided its symbol map

$$\mathbb{A}[\xi] := \sum_{k=1}^{n} \mathbb{A}_k \xi_k \colon V \to E, \qquad \xi = (\xi_1, ..., \xi_n)$$

is injective for each $\xi \in \mathbb{R}^n \setminus \{0\}$.

$\Longrightarrow$ In this situation, we have for all $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$

$$u(x) = \mathscr{F}^{-1}_{\xi \mapsto x}((\mathbb{A}^*[\xi] \circ \mathbb{A}[\xi])^{-1} \mathbb{A}^*[\xi] \widehat{\mathbb{A}[D]u}) =: \Phi(\mathbb{A}[D]u)(x)$$

and Φ is a Riesz potential operator of order $n-1$.

- Now use boundedness of singular integral operators of convolution type on L^p–spaces, $1 < p < \infty$.

Korn Operators of the First Kind

We say that $\mathbb{A}[D]$ is a **Korn(–type) operator of the first kind** if and only if

$$\forall 1 < p < \infty \, \exists C > 0 \, \forall v \in \mathrm{C}_c^\infty(\mathrm{B}; V)\colon \; \|D\varphi\|_{\mathrm{L}^1(\mathrm{B};\mathbb{R}^n \times V)} \leq C\|\mathbb{A}[D]\varphi\|_{\mathrm{L}^p(\mathrm{B};W)}$$

We then have

Theorem (G., '16)

Let $\mathbb{A}[D]$ be a standard operator. Then the following are equivalent:

- *$\mathbb{A}[D]$ is elliptic.*
- *$\mathbb{A}[D]$ is a Korn operator of the first kind.*

Korn–type Operators of the Second Type

Given a first order differential operator as above and an exponent $1 < p < \infty$, define

$$\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) := \{v \in \mathrm{L}^p(\mathrm{B};V)\colon\ \mathbb{A}[D]v \in \mathrm{L}^p(\mathrm{B};W)\}$$

and equip it with the canonical norm. We then have

Theorem (G.,'16)

Let $\mathbb{A}[D]$ be a standard differential operator of order one. Then the following are equivalent:

1. $\mathbb{A}[D]$ *has the* **FDP**, *i.e.,* $\dim(\ker(\mathbb{A}[D];\mathrm{B})) < \infty$.
2. *There exists a bounded and linear extension operator from* $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ *to* $\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$ *and* $\mathbb{A}[D]$ *is elliptic.*
3. $\mathbb{A}[D]$ *is a type–(C) operator.*
4. $\mathbb{A}[D]$ *is a Korn–type operator, i.e.,* $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \simeq \mathrm{W}^{1,p}(\mathrm{B};V)$.

Structure of the Proof

1. $\mathbb{A}[D]$ has the **FDP**, i.e., $\dim(\ker(\mathbb{A}[D];\mathrm{B})) < \infty$.
2. There exists a bounded and linear extension operator from $\mathrm{W}^{\mathbb{A},p}(\mathrm{B})$ to $\mathrm{W}^{\mathbb{A},p}(\mathbb{R}^n)$ and $\mathbb{A}[D]$ is elliptic.
3. $\mathbb{A}[D]$ is a type–(C) operator.
4. $\mathbb{A}[D]$ is a Korn–type operator, i.e., $\mathrm{W}^{\mathbb{A},p}(\mathrm{B}) \simeq \mathrm{W}^{1,p}(\mathrm{B};V)$.

$$
\begin{tikzcd}
 & 3. \arrow[d, shift right, "\text{Kalamajska '94}"'] & \\
 & 1. \arrow[u, shift right, "\text{Hol.Coupling}"'] \arrow[dr, "\text{Jones–Ext.}"] & \\
4. \arrow[ur, "\text{Peetre–Tartar–Lem.}"] & & 2. \arrow[ll]
\end{tikzcd}
.
$$

A Note on the single Steps

- On the Peetre–Tartar Lemma:

Lemma (Peetre–Tartar)

Let E_1 be a Banach and E_2, E_3 normed spaces, $A \in \mathscr{L}(E_1, E_2)$, $B \in \mathscr{L}(E_1, E_3)$ such that $\|\cdot\|_1 \approx \|A\cdot\|_2 + \|B\cdot\|_3$ and B is compact. Then $\dim(\ker(A)) < \infty$.

- On the Kalamajska Theorem: If $\mathbb{A}[D]$ is of type–(C), then for any $v \in \mathrm{C}^\infty(\Omega; V)$

$$v_i(x) = \mathfrak{P}_\omega^{l-1} v_i(x) + \sum_{j=1}^{N} \int_\Omega K_{ji}(x,y)(\mathbb{A}[D]v)_j \,\mathrm{d}y,$$

Coda

- We go back to the variational integral

$$\mathfrak{F}[v] := \int_\Omega f(\mathbb{A}[D]v)\,\mathrm{d}x.$$

with an integrand $f \in \mathrm{C}(W;\mathbb{R}_{\geq 0})$ of p–growth, $p > 1$.

- Given $u_0 \in \mathrm{W}^{1,p}(\Omega;V)$ and assuming the FDP, Dirichlet classes $\mathcal{D}_{u_0} := u_0 + \mathrm{W}_0^{1,p}(\Omega;V)$ are weakly closed.

$\longrightarrow$ Dirichlet problem has a minimiser provided $\mathfrak{F}$ is swlsc on $\mathrm{W}^{\mathbb{A},p}$ or $\mathrm{W}^{1,p}$, respectively.

$\mathscr{A}$–Quasiconvexity of Fonseca & Müller

- Given a constant–rank, linear and homogeneous first order differential operator $\mathscr{A}$ from W to Z, Fonseca & Müller proved that if $f\colon W \to \mathbb{R}_{\geq 0}$ is $\mathscr{A}$–quasiconvex, i.e.,

$$f(A) \leq \int_{\mathbb{T}^n} f(A + v(x))\,\mathrm{d}x \qquad \forall\ v \in \ker(\mathscr{A}) \cap \mathrm{C}^\infty(\mathbb{T}^n; W)) \cap \{(w)_{\mathbb{T}^n} = 0\},$$

and $v_k \rightharpoonup v$ in $\mathrm{L}^p(\Omega; W)$ together with $\mathcal{A}v_k \overset{*}{\rightharpoonup} 0$ in $\mathrm{W}^{-1,p}(\Omega; W)$, then

$$\int_\Omega f(v)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f(v_k)\,\mathrm{d}x.$$

- This gives swlsc of $\mathfrak{F}$!

Thanks for your attention!