Existence & Regularity for Functionals of Linear Growth

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Intro & Setup

- Consider functionals of the form

$$\mathfrak{F}[u] := \int_\Omega f(Du)\,\mathrm{d}x \qquad \text{among a class of functions } u\colon \Omega \to \mathbb{R}^n.$$

- Here, we assume that $f \in \mathrm{C}(\mathbb{R}^{N\times n})$ has **linear growth**, meaning

$$\exists c_1, c_2 > 0 \,\forall \mathbf{Z} \in \mathbb{R}^{N\times n}\colon\ c_1|\mathbf{Z}| \leq f(\mathbf{Z}) \leq c_2(1 + |\mathbf{Z}|)$$

- suggests to consider $\mathfrak{F}$ on

$$\mathscr{D} = u_0 + \mathrm{W}_0^{1,1}(\Omega; \mathbb{R}^N) \qquad \text{(Dirichlet)}$$

- Problem: $\mathrm{W}^{1,1}$ non–reflexive $\Rightarrow$ lack of weak compactness $\Rightarrow$ minimising sequences $(v_k) \subset \mathscr{D}$ might not possess any weakly convergent subsequences.

$\longrightarrow$ pass to the space BV of functions of bounded variation.

Relaxation

(i) Here, for $v \in \mathrm{BV}(\Omega; \mathbb{R}^N)$ with corresponding R.N.–decomposition

$$Dv = D^{ac}v + D^s v = \nabla_{ap}v\,\mathscr{L}^n\!\llcorner\Omega + \frac{\mathrm{d}Dv}{\mathrm{d}|D^s v|}|D^s v|$$

we have set

$$F_{u_0}[v] = \int_\Omega f\left(\nabla_{ap}v\right)\mathrm{d}\mathscr{L}^n + \int_\Omega f^\infty\left(\frac{\mathrm{d}Dv}{\mathrm{d}|D^s v|}\right)\mathrm{d}|D^s v|$$

$$+ \int_{\partial\Omega} f^\infty\left((\mathrm{tr}(v) - u_0) \otimes \nu_{\partial\Omega}\right)\mathrm{d}\mathcal{H}^{n-1}.$$

where $f^\infty(A) := \lim_{s\searrow 0} s f\left(\frac{A}{s}\right)$. Minima of F_{u_0} **"BV-minima"**

(ii) **"Generalised Minima"**

$$\mathcal{M}^* := \left\{ u \in \mathrm{BV}(\Omega; \mathbb{R}^N) \colon \begin{array}{c} (u_k) \subset u_0 + \mathrm{W}_0^{1,1}(\Omega)\ \mathfrak{F} - \text{minimising} \\ u_k \to u \text{ in } \mathrm{L}^1(\Omega; \mathbb{R}^N) \end{array} \right\}$$

- Fundamental Fact: (i) & (ii) lead to the same concept of minimisers!

BV– vs Generalised Minima

- **No gap:**

$$\min_{\mathrm{BV}(\Omega;\mathbb{R}^N)} F_{u_0} = \inf_{u_0+\mathrm{W}_0^{1,1}(\Omega;\mathbb{R}^N)} \mathfrak{F}.$$

- Existence of minima by direct method and weak*–compactness of UB sequences in $\mathrm{BV}(\Omega)$.

$\longrightarrow$ Under which conditions can we produce generalised minima in $\mathrm{W}^{1,1}$?

- In what follows, we shall focus on the **Neumann problem** ('natural' boundary conditions).

Regularity: μ–Ellipticity

For Sobolev regularity, we make use of the following strong convexity assumption due to BILDHAUER, FUCHS & MINGIONE:

Definition (μ–ellipticity, h–monotonicity)

A C^2–integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is called μ–**elliptic**, $1 < \mu < \infty$, provided there exist $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \frac{|\mathbf{Z}|^2}{1 + |\mathbf{Y}|^\mu} \leq \langle f''(\mathbf{Y})\mathbf{Z}, \mathbf{Z}\rangle \leq c_2 \frac{|\mathbf{Z}|^2}{1 + |\mathbf{Y}|} \qquad \text{for all } \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^{N\times n}.$$

If $(1 + |\mathbf{Y}|^\mu)^{-1}$ on the left is replaced by $h(|\mathbf{Y}|)$ with a continuous, strictly decreasing function $h\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, then we call f h–**monotone**.

Examples:

- the area integrand $\langle\cdot\rangle := \sqrt{1 + |\cdot|^2}$ is $\mu = 3$–elliptic.
- $\mu = 1$ excluded (corresponds to $L\log L$–growth)

Some History

Consider the variational integral

$$F[u] := \int_{\Omega} f(Du)\,dx \qquad \text{for } u \in \mathrm{BV}(\Omega; \mathbb{R}^N).$$

- Seregin, ($\sim$ '90s), Bildhauer & Fuchs ($\sim$ '99): The dual solution σ belongs locally to $W^{1,2}$.

- Bildhauer ($\sim$ '01):
  - $\mu = 3 \Rightarrow$ at least *one* generalised minimiser is of class $W^{1, L\log L}_{\mathrm{loc}}$.
  - $1 < \mu < 3 \Rightarrow$ at least *one* generalised minimiser is of class $W^{1,p}_{\mathrm{loc}}$ for some $1 < p < \infty$.

- Beck & Schmidt ('13): $\mu = 3 \Rightarrow$ **every** generalised minimiser is of class $W^{1, L\log L}_{\mathrm{loc}}$.

Problem: Even if f is strictly convex, f^{∞} is not.

The Neumann Problem

Consider the variational principle

to minimise $\quad \mathfrak{F}[v] := \displaystyle\int_\Omega f(|\nabla v|) - \langle T_0, \nabla v \rangle \, dx \quad$ over $W^{1,1}(\Omega; \mathbb{R}^{N \times n})$.

Under suitable regularity assumptions, the corresponding Euler–Lagrange Equation reads as

$$-\operatorname{div}\left(\frac{f'(|\nabla u|)\nabla u}{|\nabla u|}\right) = -\operatorname{div}\left(T_0\right) \qquad \text{in } \Omega,$$

$$\frac{f'(|\nabla u|)\nabla u}{|\nabla u|} \cdot \nu_{\partial\Omega} = T_0 \cdot \nu_{\partial\Omega} \qquad \text{on } \partial\Omega.$$

Again, $f \in C^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$ is assumed to have linear growth.

A Coercivity Criterion

For $\mathfrak{F}$ to be coercive on $W^{1,1} \cap \{(v)_\Omega = 0\}$, we require

$$\|T_0\|_{L^\infty} < f^\infty(1), \qquad \text{where } f^\infty(s) := \lim_{s \to \infty} \frac{f(s)}{s}. \qquad \text{(CC)}$$

Idea: First determine $R_0 = R_0(f, T) > 0$ such that

$$\frac{f(s)}{s} \geq \frac{f^\infty(1) + \|T_0\|_{L^\infty(\Omega; \mathbb{R}^{N \times n})}}{2} \qquad \text{for } s \geq R_0.$$

$$\begin{aligned}
\mathfrak{F}[w] &= \int_\Omega f(|\nabla w|) - \langle T_0, \nabla w \rangle \, dx \\
&\geq \int_{\{x \in \Omega \colon |\nabla w(x)| \geq R_0\}} \left[\frac{f^\infty(1) + \|T_0\|_{L^\infty}}{2} - \|T_0\|_{L^\infty} \right] |\nabla w| \, dx \\
&\quad - \int_{\{x \in \Omega \colon |\nabla w(x)| < R_0\}} \|T_0\|_{L^\infty} |\nabla w| \, dx \\
&\geq \frac{f^\infty(1) - \|T_0\|_{L^\infty}}{2} \|\nabla w\|_{L^1} - f^\infty(1) |\Omega| R_0.
\end{aligned}$$

A Remark on Potential Theory

$$-\operatorname{div}(A(Du)) = -\operatorname{div}(T) \qquad \text{in } \Omega$$

- Transfer properties of T to $A(Du)$ $\rightsquigarrow$ 'cancel' divergence
- E.g., if $A = \operatorname{Id}$, then $\operatorname{div}(A(Du)) = \Delta$, and then

  $$\Phi \colon T \mapsto A(Du) \text{ is a TI singular integral operator}$$

$\longrightarrow$ $\Phi \colon \mathsf{L}^p \to \mathsf{L}^p$ for $1 < p < \infty$.

- **but:** $\Phi \colon \mathsf{L}^\infty \not\to \mathsf{L}^\infty$ for $p = \infty$.
- Moreover, if A is of p–Laplace type $(1 < p < \infty)$, then

  $$T \in \mathsf{L}^\infty \Rightarrow A(Du) \simeq (1 + |Du|^2)^{\frac{p-2}{2}} Du \in \mathsf{BMO}$$

- If $p = 1$, then typically $A(\cdot) \in \mathsf{L}^\infty$ **anyway!**

Regularity for the Neumann Problem

Theorem (Beck, Bulíček, G, '17)

Let

- Ω *be a simply connected, bounded Lipschitz domain in* $\mathbb{R}^n$,
- $T_0 \in W^{2,\infty}(\Omega; \mathbb{R}^{N \times n})$ *such that (CC) holds,*
- $f \in C^2(\mathbb{R}_0^+)$ *be convex, of linear growth, and satisfy* $f(0) = f'(0) = 0$ *inducing a h-monotone variational integrand.*

Then there exists a minimiser $u \in W^{1,1}(\Omega; \mathbb{R}^N)$ *of the variational integral* $\mathfrak{F}$, *and this minimiser is unique within the class of all admissible competitor maps* $v \in W^{1,1}(\Omega; \mathbb{R}^N)$ *that satisfy* $(v)_\Omega = 0$.

Proof strategy

(i) stabilise $\mathfrak{F}_k[v] := \mathfrak{F}[v] + k^{-1}\|\nabla v\|_{\mathsf{L}^2}^2$ and obtain for each $k \in \mathbb{N}$ a minimiser $v_k \in \left(u_0 + \mathsf{W}_0^{1,2}(\Omega; \mathbb{R}^N)\right) \cap \mathsf{W}_{\mathrm{loc}}^{2,2}(\Omega; \mathbb{R}^N)$.

(ii) Chacon's biting lemma $\Rightarrow \nabla u_k \overset{b}{\rightharpoonup} E \in \mathsf{L}^1(\Omega; \mathbb{R}^{N \times n})$.

Goal: $E = \nabla v$ for some $v \in \mathsf{W}^{1,1}(\Omega; \mathbb{R}^N)$.

(iii) A–Priori Estimates $\Rightarrow \nabla u_k \to E \; \mathscr{L}^n$–a.e. in Ω & $\mathrm{curl}(E) = 0$.

(iv) Generalised Fatou's Lemma + Coercivity Criterion $\Rightarrow$ Existence.

(v) Once dealing with $\mathsf{W}^{1,1}$–functions, uniqueness is trivial by strict convexity.

Remark: The metaprinciple is that if $u \in \mathrm{BV}(\Omega; \mathbb{R}^N)$, in

$$Du = \frac{\mathrm{d}Du}{\mathrm{d}\mathscr{L}^n}\mathscr{L}^n + \frac{\mathrm{d}Du}{\mathrm{d}|D^s u|}|D^s u| = \nabla_{ap}u\mathscr{L}^n + \frac{\mathrm{d}Du}{\mathrm{d}|D^s u|}|D^s u|,$$

now $\mathrm{curl}(\nabla_{ap}u) = 0$, which is not true in general.

Remark on the proof

The most delicate part is to establish $\operatorname{curl}(E) = 0$. The idea is to estimate for test functions φ

$$\left| \int_\Omega \langle E, \operatorname{curl}(\varphi) \rangle \, dx \right| \leq \left| \int_\Omega \langle g_\ell(|E|)E, \operatorname{curl}(\varphi) \rangle \, dx \right| + \left| \int_\Omega \langle (1 - g_\ell(|E|))E, \operatorname{curl}(\varphi) \rangle \, dx \right| = \mathbf{I} + \mathbf{II}$$

where $g_\ell \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ cuts off for arguments larger than 2ℓ. For **I** use Uhlenbeck structure, **II** vanishes in the limit.

Remark: Problem in the symmetric gradient case stems from the compatibility condition

$$\operatorname{curlcurl}(E) = \Big(\sum_{i=1}^n \partial_{ik} E_i^j + \partial_{ij} E_i^k - \partial_{jk} E_i^i - \partial_{ii} E_j^k \Big)_{j,k=1,\ldots,n} = 0.$$

Thank You!

Thank you for your attention!